PaTTAN’s Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.

PDE’s Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.
Session Outline

Early numeracy concepts and skills are essential for continued achievement in mathematics. Structuring students’ earliest experiences with mathematics in a CRA sequence can help them conceptualize the concept of number and provide for more fluent and flexible counting and computation.

Objectives
- Participants will be able to model whole numbers using place value concepts.
- Participants will understand the importance of the ability to subitize and apply to skill to teach addition and subtraction.
- Participants will be able to utilize various tools (ten-frame, rek-n-rek, etc.) to model mathematical concepts.

Tech Connection

[Image of wiggio.com]

wiggio.com

group name: pattan math
password: ptnmath

Session Outline

1. Quasi-History of Math
2. Concept of Number
3. Number Bonds
4. Ten-Frames
5. Rekenrek
6. Fractions
Early Numeracy...

PA Core: Early Numbers/Operations Standards

A Quasi-History
What is this?

not factual...

A Quasi-History of Number

20,000 years ago…
- Tally Systems
- Grouping structure

(Czechoslovakia, 1937)
A Quasi-History of Number

Tally Systems
Grouping structure
(Czechoslovakia)

Place tokens in ball
Bake to prevent tampering
Mark outside with symbols to preserve records
Some time passes... local systems converge

Babylonian Number Systems c.1950 BC

A Quasi-History of Number

Chinese Number System

(BASE-10)

5 \cdot 10^2 + 1 \cdot 10^1 + 3 \cdot 10^0

500 + 10 + 3

513

A Quasi-History of Number

Germanic / Irish / Britain / Roman (Base 12)

12 troy oz. = 1 troy lb.
12 pence = 1 shilling
Dozen = 12

Gross = 12 \times 12 = 144
Great Gross = 12 \times 12 \times 12 = 1728

TIME

12 \times 2 hours = 1 day
12 months = 1 year
12 zodiac signs = Chinese Calendar

Babylon ... 60 \div 5 = 12!
“Shang-style” Counting

Language & Number

The numeric systems invented vary across time and place, and there is no doubt that the properties of such a system can facilitate or impede the development of children's mathematical understanding.

Chinese (and Asian languages based on ancient Chinese) are organized such that the numerical names are compatible with the traditional 10-base numeration system. So spoken numbers correspond exactly to their written equivalent: 15 is spoken as "ten five" and 57 as "five ten seven."

Most European systems of number words are irregular up to 100. For example in French, 92 is said as "four twenty twelve," corresponding to $4 \times 20 + 12$.

The more complicated the number word system is, the harder it is for children to learn the counting sequence.

Interpreting Numbers

1. What is this number?
2. What is the meaning of this number?

3264
Decimal (base 10)

\[3264\]

<table>
<thead>
<tr>
<th>10^3</th>
<th>10^2</th>
<th>10^1</th>
<th>10^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

\[3 \times 10^3 + 2 \times 10^2 + 6 \times 10^1 + 4 \times 10^0\]

\[3 \times 1000 + 2 \times 100 + 6 \times 10 + 4\]

Language of Number

<table>
<thead>
<tr>
<th>ABCDEFGHIJKLMNOPQRSTUVWXYZ</th>
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<tbody>
<tr>
<td>26 symbols</td>
</tr>
<tr>
<td>name \rightarrow sound \rightarrow word</td>
</tr>
<tr>
<td>Carla piece chocolate</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>0123456789</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 symbols</td>
</tr>
<tr>
<td>name \rightarrow quantity \rightarrow number</td>
</tr>
<tr>
<td>207 71 -7 0.7 1/7</td>
</tr>
</tbody>
</table>

Symbols & Meaning

- Two ways to understand letters...
  - “B” is the letter “bee” and makes the sound /b/

- What about numbers?
  - Names are taught
  - Meaning is based on place value (base 10)
Number Names & Meanings

<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>Zero</td>
<td>None</td>
<td>20</td>
<td>Twenty</td>
<td>Ten less</td>
</tr>
<tr>
<td>1</td>
<td>One</td>
<td>One</td>
<td>21</td>
<td>Twenty-one</td>
<td>Ten less, one</td>
</tr>
<tr>
<td>2</td>
<td>Two</td>
<td>Two</td>
<td>22</td>
<td>Twenty-two</td>
<td>Two less, two</td>
</tr>
<tr>
<td>3</td>
<td>Three</td>
<td>Three</td>
<td>23</td>
<td>Twenty-three</td>
<td>Two less, three</td>
</tr>
<tr>
<td>4</td>
<td>Four</td>
<td>Four</td>
<td>24</td>
<td>Twenty-four</td>
<td>Two less, four</td>
</tr>
<tr>
<td>5</td>
<td>Five</td>
<td>Five</td>
<td>25</td>
<td>Twenty-five</td>
<td>Two less, five</td>
</tr>
<tr>
<td>6</td>
<td>Six</td>
<td>Six</td>
<td>26</td>
<td>Twenty-six</td>
<td>Two less, six</td>
</tr>
<tr>
<td>7</td>
<td>Seven</td>
<td>Seven</td>
<td>27</td>
<td>Twenty-seven</td>
<td>Two less, seven</td>
</tr>
<tr>
<td>8</td>
<td>Eight</td>
<td>Eight</td>
<td>28</td>
<td>Twenty-eight</td>
<td>Two less, eight</td>
</tr>
<tr>
<td>9</td>
<td>Nine</td>
<td>Nine</td>
<td>29</td>
<td>Twenty-nine</td>
<td>Two less, nine</td>
</tr>
<tr>
<td>10</td>
<td>Ten</td>
<td>Ten</td>
<td>30</td>
<td>Thirty</td>
<td>Thirty less</td>
</tr>
<tr>
<td>11</td>
<td>Eleven</td>
<td>One ten, One</td>
<td>31</td>
<td>Thirty-one</td>
<td>Thirty less, one</td>
</tr>
<tr>
<td>12</td>
<td>Twelve</td>
<td>One ten, Two</td>
<td>32</td>
<td>Thirty-two</td>
<td>Thirty less, two</td>
</tr>
<tr>
<td>13</td>
<td>Thirteen</td>
<td>One ten, Three</td>
<td>33</td>
<td>Thirty-three</td>
<td>Thirty less, three</td>
</tr>
<tr>
<td>14</td>
<td>Fourteen</td>
<td>One ten, Four</td>
<td>34</td>
<td>Thirty-four</td>
<td>Forty less</td>
</tr>
<tr>
<td>15</td>
<td>Fifteen</td>
<td>One ten, Five</td>
<td>35</td>
<td>Forty-five</td>
<td>Forty less, five</td>
</tr>
<tr>
<td>16</td>
<td>Sixteen</td>
<td>One ten, Six</td>
<td>36</td>
<td>Fifty-six</td>
<td>Fifty less, six</td>
</tr>
<tr>
<td>17</td>
<td>Seventeen</td>
<td>One ten, Seven</td>
<td>37</td>
<td>Fifty-seven</td>
<td>Fifty less, seven</td>
</tr>
<tr>
<td>18</td>
<td>Eighteen</td>
<td>One ten, Eight</td>
<td>38</td>
<td>Fifty-eight</td>
<td>Eighty less, eight</td>
</tr>
<tr>
<td>19</td>
<td>Nineteen</td>
<td>One ten, Nine</td>
<td>39</td>
<td>Fifty-nine</td>
<td>Eighty less, nine</td>
</tr>
<tr>
<td>20</td>
<td>Twenty</td>
<td>Twenty</td>
<td>40</td>
<td>Sixty</td>
<td>Sixty less</td>
</tr>
</tbody>
</table>

What is Number Sense?

"a child's fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)

CRA Sequence of Instruction

“flexibility with mental math”
CRA

- **Concrete** (sense making by moving)
- **Representational** (sense making by drawing)
- **Abstract** (sense making with symbols)

CONSISTENT LANGUAGE

**Rationale – Doing What Works**

Research-based studies show that students who use concrete materials develop **more precise and more comprehensive mental representations**, often show more motivation and on-task behavior, understand mathematical ideas, and better apply these ideas to life situations.

(Harrison, & Harrison 1986)
(Suydam & Higgins 1977)

**Why would CRA be effective?**

- Multimodal forms of math acquisition to aid memory and retrieval
- Meaningful manipulations of materials allows students to **rationalize abstract mathematics**
- Procedural accuracy; provides **alternate to algorithm memorization**

(Winski, Riccomini, & Scheider 2008)
Other Research.

- Direct Instruction
- Errorless Teaching
- Formative Assessment
- Correct Feedback
- Improved Teacher Content Knowledge
  - Task Analyze
  - Instruct on Specific Skills or Process
  - Monitor progress
  - Correct errors

Something here...

- Multimodal forms of math acquisition to aid memory and retrieval
- Meaningful manipulations of materials allows students to rationalize abstract mathematics
- Procedural accuracy; provides alternate to algorithm memorization

Students having difficulties with math...

- Counting seen as rote, mechanical, left to right, 1:1 correspondence only; INEFFECTIVE
- Automaticity problems take up working memory, inhibit discourse & algebraic thinking

(Garett, Jordan, & Raja, 2005)
Concept of Number

“What does three really mean? What is three-ness”

What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)

What does “3” really mean?

3 three “three”

“1 ... 2 ... 3!”

“one more than 2” “one less than 4”

“is between…” “is more than…”

“is the same as…”

3 units
Teaching each symbol or Teaching the collection

Each Symbol
• Name – Meaning – Quantity
• Ability to Subitize

Collection
• Count Sequence
• Magnitude
• Missing Number
• Applications

Subitize

The ability to see a quantity and know how many, without “counting.”

Perceptual and Conceptual

Subitizing & Conceptual Counting

3 + 2 = 5
Elementary Classroom – Conceptual Addition

Teaching each symbol or Teaching the collection

Each Symbol
• Name – Meaning – Quantity
• Ability to Subitize

Collection
• Count Sequence
• Magnitude
• Missing Number
• Applications

Basic Principles of Counting

One-to-one – Counting one “thing” at a time; transfer from uncounted group to counted group (1:1 Correspondence)

Stable-order – Establishes consistent sequence

Cardinal – The last count represents the quantity in the counted group (Cardinality)

Abstraction – Applying counting to like objects, actions, sounds, etc...

Order-irrelevance – Can count in any order
What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)

From Counting to Computation

... or more efficient counting
What is the sum?

\[
\begin{align*}
1000 + 32 &= 1032 - 1 \\
999 + 32 &= 1031 \\
999 + 1 + 31 &= 1000 + 31
\end{align*}
\]

Decomposition & Compensation

**Decomposition** – decomposing numbers to compute faster
- make a 5
- make a 10
- doubles (±1)

**Compensation** – Adjust the problem to compute, then readjust the answer
- may utilize known facts.

The Mathematics Framework, Appendix F

<table>
<thead>
<tr>
<th>Count All</th>
<th>Take Away</th>
<th>Count On</th>
<th>Think +</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make 5/10</td>
<td>From 5/10</td>
<td>Make 5/10</td>
<td>From 5/10</td>
</tr>
</tbody>
</table>

Note: Today's children are taught to count down for subtractions that reach down to difficult and more obscure numbers, and then to work backward for subtractions that start higher up. In addition, children are taught how to use indicators to check their work.
Number Bonds

Composing & Decomposing Numbers

C

The Doubling Teacher

Do they "see" what I "see"?
How do I know?

Composing & Decomposing Numbers

C ➔ A
Number Bonds – Fact Families

4 = 2 + 2
4 = 1 + 3
4 = 0 + 4

Number Bonds – Fact Families

\[ a + b = c \]
\[ a + b = ? \]
\[ a + ? = c \]
\[ ? + b = c \]
\[ \{ c - a = ? \]
\[ c - ? = b \]
Partner Practice (C or R)

- Count on
- Making 5
- Making 10
- Doubles (±1)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 3</td>
<td>3 + 6</td>
</tr>
<tr>
<td>7 + 2</td>
<td>1 + 7</td>
</tr>
<tr>
<td>4 + 3</td>
<td>8 + 3</td>
</tr>
<tr>
<td>2 + 9</td>
<td></td>
</tr>
</tbody>
</table>

Partner Practice (C or R)

- Take Away
- Count on (Think +)
  - Missing addend
- Compensation
  - From 5
  - From 10
- Doubles (±1)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3 – 1</td>
<td>4 – 2</td>
</tr>
<tr>
<td>6 – 4</td>
<td>8 – 7</td>
</tr>
<tr>
<td>4 – 2</td>
<td>8 – 4</td>
</tr>
<tr>
<td>9 – 3</td>
<td></td>
</tr>
</tbody>
</table>
Ten-Frames

Decomposition

\[ 2 + 1 \]

"two and one make ..."
"two plus one makes ...
"two plus one equals ..."

Decomposition

\[ 2 + 1 = ]

\[ 2 + 1 \]
Decomposition

see the parts & see the whole

Purpose of 10-frame

• See sets of 5
• See sets of 10
• Organize in rectangular array
  • Subitize
• Reduces need to “count”
• Visually decompose numbers in sets of 5

Subitizing the 10-frame support
Ten-Frame Variations

Help students Subitize on the 10 – frame.

What do you see?

![Image with dots and numbers demonstrating addition and subtraction examples]

- $9$
- $7 + 2$
- $5 + 2 + 2$
- $9 - 2$
- $5 + 4$
- $10 - 1$

Modeling on a ten frame

![Image with dots and numbers demonstrating addition and subtraction examples]

- $6 + 2 = 8$
- $5 + 3 = 8$
- $8 - 2 = 6$

*Count on (Think +)*
Modeling on a ten frame

### 4 + 3 = 7

### 6 + 1 = 7
Doubles +1

### 8 − 1 = 7
Doubles -1

---

**Making ten**

Modeling on a ten frame

### 7 + 4 =

\[\frac{3 + 1}{3 + 1}\]

### 10 + 1 =

### 11

---

Modeling on a ten frame

### 8 + 6 =

\[\frac{2 + 4}{2 + 4}\]

### 10 + 4 =

### 14

---
Partner Practice (C)

- Count All or Take Away 8 + 9
- Counting on 4 + 6
  - Subtraction: Missing addend 7 – 3
- Making 5 3 + 4
- Making 10 7 + 8
- Doubles (±1) 12 – 4

Teaching facts w/ 10-frame support
Ten-Frame Progression

6 7
1 2
+ 3
6 3

6
+ 5
10
+ 1
11

60 + 3

Ten Frame Ideas

Rekenrek
Vocabulary

**Rekenrek** (wreck-n-wreck)

The Rekenrek (also called an arithmetic rack) has emerged as perhaps the most powerful of all models for young learners.

Developed by mathematics education researchers at the highly regarded Freudenthal Institute in the Netherlands.

Designed to reflect the natural intuitions and informal strategies that young children bring to the study of numbers, addition, and subtraction.

Provides a visual model that encourages young learners to build numbers by
- groups of five
- groups of ten
- doubling and halving strategies
- counting-on from known addition/subtraction

---

Some activities...

- **See & Slide** – Given #, make in 1 move.
- **Build a Number** – move first row, how many more on second row
- **Show Me** – Give number, make combination
- **Flash Attack** – Show beads, get number
The **Rekenrek** is a powerful tool that supports children to:

- Develop/reinforce cardinality (visualization of groupings).
- Develop one-to-one counting (organizes the count).
- Allow those who still need to count by ones to do so, but also helps children to build towards counting on.
- Visualize and build number relationships, and
- Work flexibly with numbers by encouraging decomposition strategies.

One more idea...

---
Early on in Fractions...

\[ \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \]

\[ \frac{6}{3} \]

Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \] counting \( \frac{1}{b} \)'s \\

\[ \frac{3}{3} \]

"thirds"
Vocabulary

Fraction – from Latin: *fractus*, “broken”

<table>
<thead>
<tr>
<th>numerator</th>
<th>denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>what is being counted</td>
</tr>
</tbody>
</table>

Interpreting Fractions – “counting”

<table>
<thead>
<tr>
<th>Definitions</th>
<th>Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part of whole</td>
<td>Area</td>
</tr>
<tr>
<td>Ratio</td>
<td>Length</td>
</tr>
<tr>
<td>Measurement</td>
<td>Fraction strips, Cuisenaire rods, line segments, number line</td>
</tr>
<tr>
<td>Operator/Quotient</td>
<td>Sets</td>
</tr>
<tr>
<td></td>
<td>Objects, groups or arrays</td>
</tr>
</tbody>
</table>
Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \]

**counting** \( \frac{1}{b}'s \)

\[ \frac{1}{2} = \frac{2}{4} = \frac{4}{8} \]
Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \quad \text{counting} \quad \frac{1}{b}'s \]

CRA Sequence of Instruction

"flexibility with mental math"

CRA

• **Concrete** (sense making by moving)

• **Representational** (sense making by drawing)

• **Abstract** (sense making with symbols)

CONSISTENT LANGUAGE
Resources

Tech Connection

wiggio.com

group name: pattan math
password: ptnmath

CRA Days

<table>
<thead>
<tr>
<th></th>
<th>Pittsburgh</th>
<th>Harrisburg</th>
<th>King of Prussia</th>
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<tbody>
<tr>
<td>Early Numeracy</td>
<td>10/23/15</td>
<td>10/9/15</td>
<td>10/15/15</td>
</tr>
<tr>
<td>Subtraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multiplication &amp;</td>
<td>2/25/16</td>
<td>2/16/16</td>
<td>2/25/16</td>
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<tr>
<td>Division</td>
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<td>Fractions</td>
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<td>3/18/16</td>
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<td>4/7/16</td>
<td>3/31/16</td>
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<tr>
<td>Equations</td>
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