PaTTAN’s Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.
PDE’s Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.

Session Outline

Early numeracy concepts and skills are essential for continued achievement in mathematics. Structuring students’ earliest experiences with mathematics in a CRA sequence can help them conceptualize the concept of number and provide for more fluent and flexible counting and computation.

Objectives

- Participants will be able to model whole numbers using place value concepts.
- Participants will understand the importance of the ability to subitize and apply to skill to teach addition and subtraction.
- Participants will be able to utilize various tools (ten-frame, rek-n-rek, etc.) to model mathematical concepts.
Tech Connection

wiggio.com

group name: pattan math
password: ptnmath

Session Outline

1. Quasi-History of Math
2. Concept of Number
3. Number Bonds
4. Ten-Frames
5. Rekenrek
6. Fractions
Early Numeracy...

PA Core: Early Numbers/Operations Standards
A Quasi-History

What is this?

7 X 9

not factual...
20,000 years ago…

- Tally Systems
- Grouping structure

(Czechoslovakia, 1937)
A Quasi-History of Number

Tally Systems
Grouping structure

Place tokens in ball
Bake to prevent tampering
Mark outside with symbols to preserve records

Some time passes… local systems converge

Babylonian Number Systems c.1950 BC

A Quasi-History of Number

Chinese Number System

Positional Base System

BASE-10

\[ 5 \cdot 10^2 + 1 \cdot 10^1 + 3 \cdot 10^0 \]

1000  100

500 + 10 + 3

513
**A Quasi-History of Number**

**Germanic / Irish / Britain / Roman** (Base 12)

- \(12 \text{ troy oz.} = 1 \text{ troy lb.}\)
- \(12 \text{ pence} = 1 \text{ shilling}\)
  - **Dozen** = 12
  - **Gross** = \(12 \times 12 = 144\)
  - **Great Gross** = \(12 \times 12 \times 12 = 1728\)

**TIME**

- \(12 \times 2 \text{ hours} = 1 \text{ day}\)
- \(12 \text{ months} = 1 \text{ year}\)
- 12 zodiac signs
- **Chinese Calendar**
  - *Babylon...* \(60 \div 5 = 12!\)

---

**“Shang-style” Counting**

\[ \begin{array}{c}
50 + 8 \\
5 \text{ tens}, 8
\end{array} \quad \begin{array}{c}
60 + 9 \\
6 \text{ tens}, 9
\end{array} \]
Language & Number

The numeric systems invented vary across time and place, and there is no doubt that the properties of such a system can facilitate or impede the development of children’s mathematical understanding.

Chinese (and Asian languages based on ancient Chinese) are organized such that the numerical names are compatible with the traditional 10-base numeration system. So spoken numbers correspond exactly to their written equivalent: 15 is spoken as "ten five" and 57 as "five ten seven."

Most European systems of number words are irregular up to 100. For example in French, 92 is said as "four twenty twelve," corresponding to $4 \times 20 + 12$.

The more complicated the number word system is, the harder it is for children to learn the counting sequence.

http://web.media.mit.edu/~stefanm/society/som_final.html

Interpreting Numbers

1. What is this number?

2. What is the meaning of this number?

3264
Decimal (base 10)

3264

(3 × 10^3) + (2 × 10^2) + (6 × 10^1) + (4 × 10^0)

(3 × 1000) + (2 × 100) + (6 × 10) + (4 × 1)

3000 + 200 + 60 + 4

Language of Number

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

26 symbols
name → sound → word

Carla
piece
chocolate

0 1 2 3 4 5 6 7 8 9

10 symbols
name → quantity → number

207 71 -7 0.7 1/7
Symbols & Meaning

• Two ways to understand letters…
  – “B” is the letter “bee” and makes the sound /b/

• What about numbers?
  – Names are taught
  – Meaning is based on place value (base 10)

Number Names & Meanings

<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Zero</td>
<td>None</td>
<td>20</td>
<td>Twenty</td>
<td>Two tens</td>
</tr>
<tr>
<td>1</td>
<td>One</td>
<td>One</td>
<td>21</td>
<td>Twenty-one</td>
<td>Two tens, one</td>
</tr>
<tr>
<td>2</td>
<td>Two</td>
<td>Two</td>
<td>22</td>
<td>Twenty-two</td>
<td>Two tens, two</td>
</tr>
<tr>
<td>3</td>
<td>Three</td>
<td>Three</td>
<td>23</td>
<td>Twenty-three</td>
<td>Two tens, three</td>
</tr>
<tr>
<td>4</td>
<td>Four</td>
<td>Four</td>
<td>24</td>
<td>Twenty-four</td>
<td>Two tens, four</td>
</tr>
<tr>
<td>5</td>
<td>Five</td>
<td>Five</td>
<td>25</td>
<td>Twenty-five</td>
<td>Two tens, five</td>
</tr>
<tr>
<td>6</td>
<td>Six</td>
<td>Six</td>
<td>26</td>
<td>Twenty-six</td>
<td>Two tens, six</td>
</tr>
<tr>
<td>7</td>
<td>Seven</td>
<td>Seven</td>
<td>27</td>
<td>Twenty-seven</td>
<td>Two tens, seven</td>
</tr>
<tr>
<td>8</td>
<td>Eight</td>
<td>Eight</td>
<td>28</td>
<td>Twenty-eight</td>
<td>Two tens, eight</td>
</tr>
<tr>
<td>9</td>
<td>Nine</td>
<td>Nine</td>
<td>29</td>
<td>Twenty-nine</td>
<td>Two tens, nine</td>
</tr>
<tr>
<td>10</td>
<td>Ten</td>
<td>One ten</td>
<td>30</td>
<td>Thirty</td>
<td>Three tens</td>
</tr>
<tr>
<td>11</td>
<td>Eleven</td>
<td>One ten, One</td>
<td>31</td>
<td>Thirty-one</td>
<td>Three tens, one</td>
</tr>
<tr>
<td>12</td>
<td>Twelve</td>
<td>One ten, Two</td>
<td>32</td>
<td>Thirty-two</td>
<td>Three tens, two</td>
</tr>
<tr>
<td>13</td>
<td>Thirteen</td>
<td>One ten, Three</td>
<td>33</td>
<td>Thirty-three</td>
<td>Three tens, three</td>
</tr>
<tr>
<td>14</td>
<td>Fourteen</td>
<td>One ten, Four</td>
<td>48</td>
<td>Forty-eight</td>
<td>Four tens, eight</td>
</tr>
<tr>
<td>15</td>
<td>Fifteen</td>
<td>One ten, Five</td>
<td>53</td>
<td>Fifty-three</td>
<td>Five tens, three</td>
</tr>
<tr>
<td>16</td>
<td>Sixteen</td>
<td>One ten, Six</td>
<td>62</td>
<td>Sixty-two</td>
<td>Six tens, two</td>
</tr>
<tr>
<td>17</td>
<td>Seventeen</td>
<td>One ten, Seven</td>
<td>75</td>
<td>Seventy-five</td>
<td>Seven tens, five</td>
</tr>
<tr>
<td>18</td>
<td>Eighteen</td>
<td>One ten, Eight</td>
<td>81</td>
<td>Eighty-one</td>
<td>Eight tens, one</td>
</tr>
<tr>
<td>19</td>
<td>Nineteen</td>
<td>One ten, Nine</td>
<td>99</td>
<td>Ninety-nine</td>
<td>Nine tens, nine</td>
</tr>
</tbody>
</table>

Other examples
- Forty-eight
- Fifty-three
- Sixty-two
- Seventy-five
- Eighty-one
- Ninety-nine
What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)

CRA Sequence of Instruction

“flexibility with mental math”
CRA

• Concrete (sense making by moving)

• Representational (sense making by drawing)

• Abstract (sense making with symbols)

CONSISTENT LANGUAGE

Rationale – Doing What Works

Research-based studies show that students who use concrete materials develop more precise and more comprehensive mental representations, often show more motivation and on-task behavior, understand mathematical ideas, and better apply these ideas to life situations.

(Harrison, & Harrison, 1986)
(Suydam & Higgins, 1977)
Why would CRA be effective?

- Multimodal forms of math acquisition to aid memory and retrieval
- Meaningful manipulations of materials allows students to rationalize abstract mathematics
- Procedural accuracy; provides alternate to algorithm memorization

(Reitzel, Riccomini, & Scheider, 2008)

Other Research.

- Direct Instruction
- Errorless Teaching
- Formative Assessment
- Correct Feedback
- Improved Teacher Content Knowledge
  - Task Analyze
  - Instruct on Specific Skills or Process
  - Monitor progress
  - Correct errors
Something here...

- Multimodal forms of math acquisition to aid memory and retrieval
- Meaningful manipulations of materials allows students to rationalize abstract mathematics
- Procedural accuracy; provides alternate to algorithm memorization

Students having difficulties with math...

- Counting seen as rote, mechanical, left to right, 1:1 correspondence only; INEFFECTIVE
- Automaticity problems take up working memory, inhibit discourse & algebraic thinking

(Gersten, Jordan, & Flojo, 2005)
Concept of Number

“What does three really mean? What is three-ness”

-Pennsylvania Training and Technical Assistance Network

What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)
What does “3” really mean?

3  three “three”  ● ● ●  ● ● ●

"1 ... 2 ... 3!"

“one more than 2”  “one less than 4”

“is between…”  “is more than…”

“is the same as…”  “is less than…”

3 units

Teaching each symbol or Teaching the collection

Each Symbol
• Name – Meaning – Quantity
• Ability to Subitize

Collection
• Count Sequence
• Magnitude
• Missing Number
• Applications
Subitize

The ability to see a quantity and know how many, without “counting.”

**Perceptual** and **Conceptual**

3 + 2 = 5

**Subitizing & Conceptual Counting**
Elementary Classroom – Conceptual Addition

Teaching each symbol or Teaching the collection

Each Symbol
• Name – Meaning – Quantity
• Ability to Subitize

Collection
• Count Sequence
• Magnitude
• Missing Number
• Applications
Basic Principles of Counting

**One-to-one** – Counting one “thing” at a time; transfer from uncounted group to counted group (1:1 Correspondance)

**Stable-order** – Establishes consistent sequence

**Cardinal** – The last count represent the quantity in the counted group (Cardinality)

**Abstraction** – applying counting to like objects, actions, sounds, etc…

**Order-irrelevance** – Can count in any order

---

Stages of Early Arithmetical Learning

<table>
<thead>
<tr>
<th>Stage</th>
<th>Indicators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 0: Emergent Counting</td>
<td>Cannot count visible items. The child may not know the number words. The child cannot coordinate number words with items.</td>
</tr>
<tr>
<td>Stage 1: Perceptual Counting</td>
<td>Can count perceived items. May involve seeing, hearing or feeling items.</td>
</tr>
<tr>
<td>Stage 2: Figurative Counting</td>
<td>Can count the total of two collections. Counts from one.</td>
</tr>
<tr>
<td>Stage 3: Initial Number Sequence</td>
<td>Child uses and understands counting-on rather than counting-from-one. Uses counting on to solve addition and missing addend tasks. May use count-down-from strategies.</td>
</tr>
<tr>
<td>Stage 4: Intermediate Number Sequence</td>
<td>The child uses and understands: • count-down-from strategies • count-down-to strategies The child can choose the most efficient strategy.</td>
</tr>
<tr>
<td>Stage 5: Facile Number Sequence</td>
<td>The child uses a range of non-count by one strategies: • Compensation • Using known results • Adding to ten • Commutativity • Subtraction as the inverse of addition • Awareness of ten as a teen number</td>
</tr>
</tbody>
</table>

What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)

From Counting to Computation

... or more efficient counting
**What is the sum?**

\[
1000 + 32 = 1032 - 1
\]

\[
999 + 32 = 1031
\]

\[
999 + 1 + 31 = 1000 + 31
\]

**Decomposition & Compensation**

**Decomposition** – decomposing numbers to compute faster
- make a 5
- make a 10
- doubles (±1)

**Compensation** – Adjust the problem to compute, then readjust the answer
- may utilize known facts.

**Strategies**

**Compensation**

**Deformation**

**Strategies**
### Number Bonds

<table>
<thead>
<tr>
<th>Level 1: Count all</th>
<th>8 + 6 = 14</th>
<th>14 - 8 = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count All</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Take Away</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 2: Count on</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Count On</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Think +</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 3: Recompose</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Make 5/10</td>
<td></td>
<td>From 5/10</td>
</tr>
<tr>
<td>Doubles = n</td>
<td>6 + 8</td>
<td>= 6 + 6 + 2</td>
</tr>
</tbody>
</table>

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.
Composing & Decomposing Numbers

The Doubting Teacher

Do they “see” what I “see”? How do I know?

Composing & Decomposing Numbers

Decomposing Numbers

Working With Number Bonds!
Number Bonds – Fact Families

\[ 4 = 2 + 2 \]
\[ 4 = 1 + 3 \]
\[ 4 = 0 + 4 \]

Number Bonds – Fact Families

\[ a + b = c \]
\[ a + b = ? \]
\[ a + ? = c \]
\[ ? + b = c \]
\[ \{ c - a = ? \]  
\[ c - ? = b \]
Concrete/Representational Modeling

Partner Practice (C or R)

- Count on 2 + 3
- Making 5 3 + 6
- Making 10 7 + 2
- Doubles (±1) 1 + 7

The Doubling Teacher
Do they “see” what I “see”? How do I know?

2 + 9
Concrete/Representational Modeling

**Partner Practice (C or R)**

- Take Away
- Count on (Think +)
  - Missing addend
- Compensation
  - From 5
  - From 10
- Doubles (±1)

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 - 1</td>
<td></td>
</tr>
<tr>
<td>4 - 2</td>
<td></td>
</tr>
<tr>
<td>6 - 4</td>
<td></td>
</tr>
<tr>
<td>8 - 7</td>
<td></td>
</tr>
<tr>
<td>4 - 2</td>
<td></td>
</tr>
<tr>
<td>8 - 4</td>
<td></td>
</tr>
<tr>
<td>9 - 3</td>
<td></td>
</tr>
</tbody>
</table>
Ten-Frames

Decomposition
Decomposition

\[
2 + 1 = 2 + 1
\]

"two and one make ...
"two plus one makes ...
"two plus one equals ...

Decomposition

\[
\text{see the parts & see the whole}
\]
Purpose of 10-frame

- See sets of 5
- See sets of 10
- Organize in rectangular array
  - Subitize
- Reduces need to “count”
- Visually decompose numbers in sets of 5

Subitizing the 10-frame support
Ten-Frame Variations

Help students Subitize on the 10 – frame.

What do you see?

5 + 2 + 2 = 9
7 + 2 = 9
5 + 4 = 10 – 1
9 – 2 = 7 + 2
Modeling on a ten frame

<table>
<thead>
<tr>
<th>Count on</th>
<th>6 + 2 = 8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5 + 3 = 8</td>
</tr>
<tr>
<td>Make 5</td>
<td>8 − 2 = 6</td>
</tr>
<tr>
<td></td>
<td>(Think +)</td>
</tr>
</tbody>
</table>

Modeling on a ten frame

<table>
<thead>
<tr>
<th>4 + 3 = 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 + 1 = 7</td>
</tr>
<tr>
<td>Doubles +1</td>
</tr>
<tr>
<td>8 − 1 = 7</td>
</tr>
<tr>
<td>Doubles -1</td>
</tr>
</tbody>
</table>
Modeling on a ten frame  

**Making ten**

\[ \begin{align*} 
7 + 4 &= 3 + 1 \\
10 + 1 &= 11 
\end{align*} \]

Modeling on a ten frame  

**Making ten**

\[ \begin{align*} 
8 + 6 &= 2 + 4 \\
10 + 4 &= 14 
\end{align*} \]
## Concrete Modeling

**Partner Practice (C)**

- Count All or Take Away
- Counting on
  - Subtraction: Missing addend
- Making 5
- Making 10
- Doubles (±1)

<table>
<thead>
<tr>
<th>Operation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8 + 9$</td>
<td></td>
</tr>
<tr>
<td>$4 + 6$</td>
<td></td>
</tr>
<tr>
<td>$7 - 3$</td>
<td></td>
</tr>
<tr>
<td>$3 + 4$</td>
<td></td>
</tr>
<tr>
<td>$7 + 8$</td>
<td></td>
</tr>
<tr>
<td>$12 - 4$</td>
<td></td>
</tr>
</tbody>
</table>
Teaching facts w/ 10-frame support

Ten-Frame Progression

\[ \begin{array}{c}
1 \\
2 \\
7 \\
+ \\
3 \\
6 \\
63 \\
\end{array} \]

\[ \begin{array}{c}
6 \\
+ \\
5 \\
10 \\
+ \\
1 \\
11 \\
\end{array} \]

\[ \begin{array}{c}
60 \\
+ \\
3 \\
\end{array} \]
Ten Frame Ideas

Rekenrek

Pennsylvania Training and Technical Assistance Network
The Rekenrek (also called an arithmetic rack) has emerged as perhaps the most powerful of all models for young learners.

Developed by mathematics education researchers at the highly regarded Freudenthal Institute in the Netherlands.

Designed to reflect the natural intuitions and informal strategies that young children bring to the study of numbers, addition, and subtraction.

Provides a visual model that encourages young learners to build numbers by
- groups of five
- groups of ten
- doubling and halving strategies
- counting-on from known addition/subtraction
Some activities…

- **See & Slide** – Given #, make in 1 move.
- **Build a Number** – move first row, how many more on second row
- **Show Me** – Give number, make combination
- **Flash Attack** – Show beads, get number

---

**THE REKENREK**

The *Rekenrek* is a powerful tool that supports children to
- develop/reinforce cardinality (visualization of groupings),
- develop one-to-one counting (organizes the count),
- allows those who still need to count by ones to do so, but also helps children to build towards counting on,
- visualize and build number relationships, and
- work flexibly with numbers by encouraging decomposition strategies.
Rekenrek

One more idea...
Early on in Fractions…
Early on in Fractions...

\[
\frac{a}{b} = a \times \frac{1}{b} \quad \text{counting "} \frac{1}{b}\text{"'s "}
\]

Vocabulary

**Fraction** – from Latin: *fractus*, “broken”

- **Numerator**
- **Denominator**
- **Count**
- **What is being counted**
# Interpreting Fractions – “counting”

<table>
<thead>
<tr>
<th>Definitions</th>
<th>Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part of whole</td>
<td>Area</td>
</tr>
<tr>
<td>Ratio</td>
<td>circles, pattern blocks, graph/dot paper</td>
</tr>
<tr>
<td></td>
<td>paper, paper folding</td>
</tr>
<tr>
<td>Measurement</td>
<td>Length</td>
</tr>
<tr>
<td>Operator/Quotient</td>
<td>Fraction strips, Cuisenaire rods, line</td>
</tr>
<tr>
<td></td>
<td>segments, number line</td>
</tr>
</tbody>
</table>

**Sets**
- Objects, groups or arrays

**VIDEO**
What & When?

Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \]

"counting" \( \frac{1}{b} \)'s "

\[ \frac{1}{2} = \frac{2}{4} = \frac{4}{8} \]
Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \quad \text{counting} \quad \frac{1}{b}'s \]

\[ \frac{1}{2} = \frac{2}{4} = \frac{4}{8} \]

Early on in Fractions...

\[ \frac{a}{b} = a \times \frac{1}{b} \quad \text{counting} \quad \frac{1}{b}'s \]

\[ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, ... \]
CRA Sequence of Instruction

“flexibility with mental math”

CRA

• **Concrete** (sense making by moving)

• **Representational** (sense making by drawing)

• **Abstract** (sense making with symbols)

**CONSISTENT LANGUAGE**
Resources

Tech Connection

wiggio.com

group name: pattan math
password: ptnmath
## CRA Days

<table>
<thead>
<tr>
<th></th>
<th>Pittsburgh</th>
<th>Harrisburg</th>
<th>King of Prussia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early Numeracy</td>
<td>10/23/15</td>
<td>10/9/15</td>
<td>10/15/15</td>
</tr>
<tr>
<td>Multiplication &amp; Division</td>
<td>2/25/16</td>
<td>2/16/16</td>
<td>2/25/16</td>
</tr>
<tr>
<td>Fractions</td>
<td>3/15/16</td>
<td>3/16/16</td>
<td>3/18/16</td>
</tr>
<tr>
<td>Integers &amp; Equations</td>
<td>4/7/16</td>
<td>3/31/16</td>
<td>4/1/16</td>
</tr>
</tbody>
</table>

www.pattan.net

## Contact Information

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Commonwealth of Pennsylvania

Tom Wolf, Governor