Mathematics in Autism Interventions

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Pennsylvania Training and Technical Assistance Network
PaπAN on twitter

#NAC22

#ptnmath
PaTTAN’s Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.

PDE’s Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.
You are the expert with your students!
Session Outline

Quantitative verbal concepts begin with the ability to tact quantities and extends across exemplars. This session will introduce participants to fundamental verbal concepts in mathematics. Participants will be able utilize a skills sequence to teach these concepts.

Objectives

- Participants will extend their understanding of teaching students to Tact items to build students’ conceptual knowledge
- Participants will be able determine apply quantitative concepts across mathematical domains
- Participants will be able to identify multiple exemplars for mathematical concepts
- Participants will understand the progression of skill sequence when using the CRA model of instruction
Session Outline

1. Application of ABA and Verbal Behavior to mathematic Instruction
2. Concept of Number
3. Two-way Quantitative Verbal Concepts
4. One-way Quantitative Verbal Concepts
“But counting is so simple, I know he can learn it”
Teaching each symbol or Teaching the collection

Each Symbol
• Name – Meaning – Quantity
• Ability to Subitize

Collection
• Counting
• Magnitude
• Applications
5 Strands of Mathematical Proficiency

- Conceptual Understanding
  - Strategic Competence
  - Productive Disposition
  - Adaptive Reasoning
  - Procedural Fluency

Prerequisites → Conceptual Understanding → Math Topic

(NRC, 2001)
What is conceptual understanding?

Extended Tacts

- **Generalization** must occur
  - Can apply to **novel items** without explicit teaching
  - Across...
    1. People
    2. Places
    3. Materials
    4. Instructions
    5. Time

- **Feature/Function/Class**
  - Tacting **critical features** may facilitate concept acquisition

- The tact is involved in the process of **joint control** which assists students in effective **verbal recall** and effective **listener responding**
What is **conceptual understanding?**

**Atomic Repertoires**
- New combination of skills applied to new behaviors
- Most of our spoken language is a result of ARs

*What are the prerequisite skills needed for the atomic repertoires for the math content?*
- Imitation
- Echoic
- Tacts
- Textual Behavior (reading texts/symbols)
- Transcriptive Behavior (copying text/symbols)
- Etc…

**We must identify and teach the skills!**
It is important for students to be able to “read” mathematics.

However, textual behavior is only relevant when students understand the meaning of the words.

OR

Interpreting math symbols is only relevant when they understand their meaning.
Quantitative Verbal Concepts

“Language used to describe something that is connected to a value (numerical/spatial)”

More later…….
Now let’s talk math!
Concept of Number

“What does three really mean?
What is three-ness”

-MM
What does “3” really mean?

3 three “three”

"1 ... 2 ... 3!"

“one more than 2” “one less than 4”

“is between… ” “is more than… ”

“is the same as… ” “is less than… ”

3 units
What is Number Sense?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)
### Concept Matrix

<table>
<thead>
<tr>
<th>Teacher (antecedent)</th>
<th>Find digit</th>
<th>Write digit</th>
<th>Write text</th>
<th>Say number</th>
<th>Make/select pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>Say number</td>
<td>LR</td>
<td>Trans.</td>
<td>Trans.</td>
<td>Echoic</td>
<td>LR</td>
</tr>
<tr>
<td>Show digit</td>
<td>MtS</td>
<td>Trans.</td>
<td>IV</td>
<td>IV</td>
<td>MtS</td>
</tr>
<tr>
<td>Show text</td>
<td>MtS</td>
<td>IV</td>
<td>Trans.</td>
<td>Text</td>
<td>MtS</td>
</tr>
<tr>
<td>Show pattern</td>
<td>MtS</td>
<td>IV</td>
<td>IV</td>
<td>Tact</td>
<td>MtS</td>
</tr>
</tbody>
</table>
CAUTION!

• Before you begin math instruction:
  – Can student say (echo) necessary responses?
  – Do they already have a broad tact repertoire?
  – Are they not making conditional discrimination errors?
  – Are they mediating their response when problem solving (answering questions, following multiple component directions)?
  – Can they write necessary responses?
Symbols & Meaning

• Two ways to understand letters…
  – “B” is the letter “bee” and makes the sound /b/

• What about numbers?
  – Names are taught
  – Meaning is based on place value (base 10)
<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Zero</td>
<td>None</td>
<td>20</td>
<td>Twenty</td>
<td>Two tens</td>
</tr>
<tr>
<td>1</td>
<td>One</td>
<td>One</td>
<td>21</td>
<td>Twenty-one</td>
<td>Two tens, one</td>
</tr>
<tr>
<td>2</td>
<td>Two</td>
<td>Two</td>
<td>22</td>
<td>Twenty-two</td>
<td>Two tens, two</td>
</tr>
<tr>
<td>3</td>
<td>Three</td>
<td>Three</td>
<td>23</td>
<td>Twenty-three</td>
<td>Two tens, three</td>
</tr>
<tr>
<td>4</td>
<td>Four</td>
<td>Four</td>
<td>24</td>
<td>Twenty-four</td>
<td>Two tens, four</td>
</tr>
<tr>
<td>5</td>
<td>Five</td>
<td>Five</td>
<td>25</td>
<td>Twenty-five</td>
<td>Two tens, five</td>
</tr>
<tr>
<td>6</td>
<td>Six</td>
<td>Six</td>
<td>26</td>
<td>Twenty-six</td>
<td>Two tens, six</td>
</tr>
<tr>
<td>7</td>
<td>Seven</td>
<td>Seven</td>
<td>27</td>
<td>Twenty-seven</td>
<td>Two tens, seven</td>
</tr>
<tr>
<td>8</td>
<td>Eight</td>
<td>Eight</td>
<td>28</td>
<td>Twenty-eight</td>
<td>Two tens, eight</td>
</tr>
<tr>
<td>9</td>
<td>Nine</td>
<td>Nine</td>
<td>29</td>
<td>Twenty-nine</td>
<td>Two tens, nine</td>
</tr>
<tr>
<td>10</td>
<td>Ten</td>
<td>One ten</td>
<td>30</td>
<td>Thirty</td>
<td>Three tens</td>
</tr>
<tr>
<td>11</td>
<td>Eleven</td>
<td>One ten, One</td>
<td>31</td>
<td>Thirty-one</td>
<td>Three tens, one</td>
</tr>
<tr>
<td>12</td>
<td>Twelve</td>
<td>One ten, Two</td>
<td>32</td>
<td>Thirty-two</td>
<td>Three tens, two</td>
</tr>
<tr>
<td>13</td>
<td>Thirteen</td>
<td>One ten, Three</td>
<td>48</td>
<td>Forty-eight</td>
<td>Four tens, eight</td>
</tr>
<tr>
<td>14</td>
<td>Fourteen</td>
<td>One ten, Four</td>
<td>53</td>
<td>Fifty-three</td>
<td>Five tens, three</td>
</tr>
<tr>
<td>15</td>
<td>Fifteen</td>
<td>One ten, Five</td>
<td>62</td>
<td>Sixty-two</td>
<td>Six tens, two</td>
</tr>
<tr>
<td>16</td>
<td>Sixteen</td>
<td>One ten, Six</td>
<td>75</td>
<td>Seventy-five</td>
<td>Seven tens, five</td>
</tr>
<tr>
<td>17</td>
<td>Seventeen</td>
<td>One ten, Seven</td>
<td>81</td>
<td>Eighty-one</td>
<td>Eights tens, one</td>
</tr>
<tr>
<td>18</td>
<td>Eighteen</td>
<td>One ten, Eight</td>
<td>99</td>
<td>Ninety-nine</td>
<td>Nine tens, nine</td>
</tr>
<tr>
<td>19</td>
<td>Nineteen</td>
<td>One ten, Nine</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Other examples:

<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>Forty-eight</td>
<td>Four tens, eight</td>
</tr>
<tr>
<td>53</td>
<td>Fifty-three</td>
<td>Five tens, three</td>
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Teaching each symbol or Teaching the collection

Each Symbol
- Name – Meaning – Quantity
- Ability to Subitize

Collection
- Counting
- Magnitude
- Applications
Subitize

The ability to see a quantity and know how many, without “counting.”

Perceptual and Conceptual
Subitization

Research indicated that dice patterns and rectangular arrays are the easiest for students to learn.

Don’t go crazy!

Subitization
Subitization – Tacting a Feature

Verbal Conditional Discrimination must be established.

• What is it?
• What part is it?
• How many?
## Subitization – Tacting a Feature

<table>
<thead>
<tr>
<th>Trial</th>
<th>Teacher</th>
<th>Learner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tact Prompt for Feature</td>
<td>Presents item “How many? Six.”</td>
<td>“Six”</td>
</tr>
<tr>
<td>Tact Transfer</td>
<td>“How many?”</td>
<td>“Six”</td>
</tr>
<tr>
<td>Distractor(s)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Tact Trial Item</td>
<td>Presents item “What are these?”</td>
<td>“Red-veined Dropwing Dragonflies”</td>
</tr>
<tr>
<td>Tact Feature Check</td>
<td>Presents item “How many?”</td>
<td>“Six”</td>
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**Error Correction** – Run a contrast correction as part of the distract trial sequence
# Subitization – Tacting a Feature

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<td>?</td>
</tr>
<tr>
<td>Tact Feature Check</td>
<td>Presents item “How many?”</td>
<td>“Six”</td>
</tr>
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</table>
Generalization & discrimination should be present for the items in the set.

The concept of quantity has been developed when the individual can subitize (tact) the number of novel items in a set without explicit training.
What is CRA?
CRA Sequence of Instruction

1. Introduce the mathematical concept(s)
2. Teach and practice modeling procedures concretely
3. Connect the concrete to a representation of the concrete
4. Practice modeling the procedure representationally
5. Connect the representation to the abstract symbols
6. Practice the abstract modeling of the procedure
7. Make connections between all three models to help students monitor their thinking and choice of representation
8. Provide opportunities for student choice.

“flexibility with mental math”
CRA

- **Concrete** (sense making by moving)
- **Representational** (sense making by drawing)
- **Abstract** (sense making with symbols)

**CONSISTENT LANGUAGE**
Rationale – Doing What Works

Research-based studies show that students who use concrete materials develop more precise and more comprehensive mental representations, often show more motivation and on-task behavior, understand mathematical ideas, and better apply these ideas to life situations.

(Harrison, & Harrison, 1986)
(Suydam & Higgins, 1977)
Why would CRA be effective?

- Multiple responses for each mathematical concept to aid “memory and retrieval”

- Meaningful manipulations of materials allows students to engage in overt responses to assist in problem solving abstract mathematics

- Procedural accuracy; provides alternate to algorithm memorization

(Witzel, Riccomini, & Scheider, 2008)
Other Research.

- Direct Instruction
- Errorless Teaching
- Formative Assessment
- Correct Feedback
- Improved Teacher Content Knowledge
  - Task Analyze
  - Instruct on Specific Skills or Process
  - Monitor progress
  - Correct errors
Students having difficulties with math...

• Counting seen as rote, mechanical, left to right, 1:1 correspondence only; INEFFECTIVE

• Automaticity problems interfere with concept formation and problem solving

(Gersten, Jordan, & Flojo, 2005)
Early Quantitative Concepts:

Five- & Ten-Frames

“What does three really mean? What is three-ness”

-MM
Decomposition

see the parts & see the whole
Purpose of 10-frame

- See sets of 5
- See sets of 10
- Organize in rectangular array
  - Subitize
- Reduces need to “count”
- Visually decompose numbers in sets of 5
Subitization – Ten Frames
Subitization – Ten Frames
Subitization – Ten Frames
Five Counting Principles

1. **Stable-order** – Establishes consistent sequence

2. **Cardinal** – The last count represents the quantity in the counted group (*Cardinality*)

3. **One-to-one** – Counting one “thing” at a time; transfer from uncounted group to counted group (*1:1 Correspondance*)

4/5. **Order-irrelevance** – Can count in any order

**Abstraction** – applying counting to like objects, actions, sounds, etc…
Early Instructional Sequence
Suggested Sequence

- TARGET SKILLS 1-6: Tact Typical Dice Pattern
- TARGET SKILLS 7-12: LR – Select Typical Dice Pattern (will likely get for free with teaching tact)
- TARGET SKILLS 13-18: Build Dice Pattern (start as soon as possible once initial patterns acquired as tact/LR)
- TARGET SKILLS 19-23: Tact Various Atypical Dice Pattern (on card)
- TARGET SKILLS 24-28: Tact Various Atypical Dice Pattern (with objects)
- TARGET SKILLS 29-33: Tact Dice Pattern (on card) in discrimination
- TARGET SKILLS 34-38: Tact Dice Pattern (with objects) in discrimination
Suggested Sequence

- TARGET SKILLS 39-44: Copy written numbers
- TARGET SKILLS 45-50: Write upon dictation
- TARGET SKILLS 51-56: Tact Numeral
- TARGET SKILLS 57-62: LR Select Numeral
- TARGET SKILLS 63-68: Tact Solid Pattern on 10 frame with objects
- TARGET SKILLS 69-73: Tact Solid Pattern on 10 frame with objects in discrimination
- TARGET SKILLS 74-79: Tact Solid Pattern on 10 frame (on card)
- TARGET SKILLS 80-84: Tact Pattern on 10 frame (on card) in discrimination
Suggested Sequence

- TARGET SKILLS 85-89: Build 10 frame with objects (1-5)
- TARGET SKILLS 90-93: Tact Quantity of Each Item & Altogether on Dice Pattern
- TARGET SKILLS 94-97: Tact Quantity of Each Item & Altogether on 10 frame Pattern
- TARGET SKILLS 98-102: Draw 10 frame (1-5)
- TARGET SKILLS 103-108: Tact and Select symbols (+ - =)
- TARGET SKILLS 109-113: Conceptual addition with 10 frames (up to 5)
- TARGET SKILLS 114-118: Tact Solid Pattern on 10 frame with objects (6-10)
- TARGET SKILLS 119-123: Tact Solid Pattern on 10 frame (on card) (6-10)
Suggested Sequence

• TARGET SKILLS 124-128: Tact Solid Pattern on 10 frame on card (6-10)
• TARGET SKILLS 129-133: Tact Pattern on 10 frame (on card) in discrimination (6-9)
• TARGET SKILLS 134-138: Build 10 frame with objects (6-10)
• TARGET SKILLS 139-143: Draw 10 frame (6-10)
• TARGET SKILLS 144-148: Tact Numeral
• TARGET SKILLS 149-153: LR Select Numeral
• TARGET SKILLS 154-158: Copy Written Word
• TARGET SKILLS 159-163: Write Upon Dictation
• TARGET SKILLS 164-173: Count to ____
Suggested Sequence

• TARGET SKILLS 174-184: Read number Word
• TARGET SKILLS 185-195: Selects number Word
• TARGET SKILLS 196-206: Copy Written Word
• TARGET SKILLS 207-217: Write Upon Dictation
• TARGET SKILLS 218-227: 1:1 Correspondence Count to:
• TARGET SKILLS 228-236: Give Specified Number of Objects from Larger Set
Understanding Individual Quantities

Responding fluently to a wide range of exemplars (materials, arrangement, etc…) of quantity across operants

✓ Prerequisites
  • Level 3 students on VBMAPP, with strong verbal conditional discrimination
  • Ensure/strengthen Echoic/Imitation

Quantitative Patterns
1. Solid Dice Patterns (0-5)
   • Select and build solid patterns
2. Dice Patterns in Discrimination (0-5 + other)
3. Dice Patterns in Discrimination (0-5, within)
4. Solid Ten Frame Patterns (0-5)
   • Select and build solid patterns
5. Ten Frame Patterns in Discrimination (0-5)
6. Ten Frame Patterns in Discrimination (0-5 + others)
3 + 2 = 5
Subitizing & Conceptual Counting/Addition

Responding fluently to a wide range of exemplars (materials, arrangement, etc...) of quantity across operants

✓ Prerequisites
  • Level 3 students on VBMAPP, with strong verbal conditional discrimination
  • Ensure/strengthen Echoic/Imitation

Sets/Subsets and Digits (“Conceptual Addition”)

1. Addition (+) as “put together”
2. Equal (=) as “same value”
3. Use symbols with manipulatives (horz+vert)
   • In place and as model
   • One color per problem, but vary
   • Two colors
   • Random colors
   • Other objects, same then random (i.e. generalization)
4. Interpret digits to build/select subsets and determine set
Quantitative Verbal Concepts
Quantitative Verbal Concepts (QVC)

“Language used to describe something that is connected to a value (numerical/spatial)”

“two-way”
- more/less
- full/empty
- wide/narrow
- most/least
- long/short

Lead to...

“one-way”
- minimum
- greater than
- volume
- area
- equality
Two-way QVCs

“Language used to describe something that is connected to a value (numerical/spatial)"

more/less
full/empty
wide/narrow
most/least
long/short
# Two-way QVC – Tacting Adjectives

<table>
<thead>
<tr>
<th>Trial</th>
<th>Teacher</th>
<th>Learner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tact Trial with echoic prompt</td>
<td>Presents dice patterns of 5 and 2, identical except for attribute.</td>
<td>“More”</td>
</tr>
<tr>
<td></td>
<td>“Let’s talk about more and less.” Point to 5. “This one is… more.”</td>
<td></td>
</tr>
<tr>
<td>Tact Transfer</td>
<td>“This one is…”</td>
<td>“More”</td>
</tr>
<tr>
<td>Tact Trial with echoic prompt</td>
<td>Point to 2. “This one is… less.”</td>
<td>“Less”</td>
</tr>
<tr>
<td>Tact Transfer</td>
<td>“This one is…”</td>
<td>“Less”</td>
</tr>
<tr>
<td>Distractor(s)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Tact Check</td>
<td>Point to 2. “This one is…”</td>
<td>“Less”</td>
</tr>
<tr>
<td>Distractor(s)</td>
<td>? (include “How many?”)</td>
<td>?</td>
</tr>
<tr>
<td>Tact Check</td>
<td>Point to 5. “This one is…”</td>
<td>“More”</td>
</tr>
</tbody>
</table>

Continue with tact checks. Vary tact checks with distractors as shown above. End with the below tact check.

| Tact Check | “What are these?” | “Planes” |
Two-way QVC – Tacting Adjectives

<table>
<thead>
<tr>
<th>Trial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tact Trial with echoic prompt</td>
</tr>
<tr>
<td>Tact Transfer</td>
</tr>
<tr>
<td>Tact Trial with echoic prompt</td>
</tr>
<tr>
<td>Tact Transfer</td>
</tr>
<tr>
<td>Distractor(s)</td>
</tr>
<tr>
<td>Tact Check</td>
</tr>
<tr>
<td>Distractor(s)</td>
</tr>
<tr>
<td>Tact Check</td>
</tr>
<tr>
<td>Continue with tact checks. Vary tact checks with distractors as shown above. End with the below tact check.</td>
</tr>
<tr>
<td>Tact Check item</td>
</tr>
</tbody>
</table>
Two-way QVC – Data Collection

<table>
<thead>
<tr>
<th>Target: more/less</th>
<th>Date introduced</th>
<th>Date Mastered</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identical Sets:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single Digits</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dice Patterns</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ten Frames</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Novel Identical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trucks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mixed Mastered pairs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Known with one novel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NET</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intraverbal Opposites:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# One-way QVCs

"Language used to describe something that is connected to a value (numerical/spatial)"

<table>
<thead>
<tr>
<th>Term</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimum</td>
<td>What are the 2-way QVCs (adjectives) related to this?</td>
</tr>
<tr>
<td>greater than</td>
<td>What are the other prerequisites?</td>
</tr>
<tr>
<td>volume</td>
<td></td>
</tr>
<tr>
<td>area</td>
<td></td>
</tr>
<tr>
<td>equality</td>
<td></td>
</tr>
<tr>
<td>fraction</td>
<td>How is this concept generalized?</td>
</tr>
</tbody>
</table>
“A limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation.”


“Students who understand the equal sign as a relational symbol of equivalence are more successful solving algebraic equations than their peers who do not have such an understanding.”

**Equality Misconception(s)**

- Children see $=$ as a “do this” command or “complete the operations”
- Children should be able to also see $=$ as “is the same as”

\[
\begin{align*}
2 + 3 &= ? \\
5 - 3 &= ? \\
5 - 2 &= ?
\end{align*}
\]

Do the operation…

\[
\begin{align*}
9 + 3 &= 12 + 6 \\
9 + 3 &= 12 + 6 = 18 \\
9 + 3 &= 18 + 6
\end{align*}
\]

“9 plus 3 is 12; plus 6.”

“9 plus 3 is the same as 12 plus 6.”

“9 plus 3 is the same as 6 plus 6.”

\[
9 + 3 = 6 + 6
\]

---


Equality

\[ 3 = 2 + 1 = 1 + 2 \]
Equality

3 + 2 = 5

4 + 1 = 5
Magnitude on Number Line

Touch 4.
Extend more/less.

“Which side is more?”
“Touch 7. Is 7 more or less than 4?”
“Which side is greater than (less than) 4 (#)?”
“Is 7 (#) greater than (less than) 4(#)?”

$x > 4$

$x < 4$
Teach the blocks!
Concrete – Representation – Abstract

The Doubting Teacher

Do they “see” what I “see”? How do I know?
### Place Value (expanded form)

<table>
<thead>
<tr>
<th>Concrete</th>
<th>Representational</th>
<th>Abstract</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Concrete" /></td>
<td><img src="image2" alt="Representational" /></td>
<td>243; 2 hundreds 200</td>
<td>How many hundreds? What is the value?</td>
</tr>
<tr>
<td><img src="image3" alt="Concrete" /></td>
<td><img src="image4" alt="Representational" /></td>
<td>243; 4 tens 40</td>
<td>How many tens? What is the value?</td>
</tr>
<tr>
<td><img src="image5" alt="Concrete" /></td>
<td><img src="image6" alt="Representational" /></td>
<td>243; 3 ones 3</td>
<td>How many ones? What is the value?</td>
</tr>
</tbody>
</table>

### Reread the problem.

200 + 40 + 3

How do we write the number in expanded form?

"two hundred forty three"

How do we read the number?

### May support the Abstract portions with place value cards

![Place Value Cards](image7)
Addition
<table>
<thead>
<tr>
<th>Concrete</th>
<th>Representational</th>
<th>Abstract</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Concrete" /></td>
<td><img src="image2" alt="Representational" /></td>
<td>243</td>
<td>How many ones does each number have?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+139</td>
<td>“3 ones and 9 ones”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>How many ones does the sum have?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>243+139</td>
<td>“12 ones.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>Can I regroup 10 ones to make a ten?</td>
</tr>
<tr>
<td><img src="image3" alt="Concrete" /></td>
<td><img src="image4" alt="Representational" /></td>
<td>1</td>
<td>“Yes, 1 ten and 2 ones.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>243+139</td>
<td>How many tens does each number have?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>82</td>
<td>“1 ten, 4 tens, and 3 tens.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>How many tens does the sum have?</td>
</tr>
<tr>
<td><img src="image5" alt="Concrete" /></td>
<td><img src="image6" alt="Representational" /></td>
<td>1</td>
<td>“8 tens.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>243+139</td>
<td>Can I regroup 10 tens to make a hundred?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“No.”</td>
</tr>
<tr>
<td><img src="image7" alt="Concrete" /></td>
<td><img src="image8" alt="Representational" /></td>
<td>1</td>
<td>How many hundreds does each number have?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>243+139</td>
<td>“2 hundreds and 1 hundred.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>382</td>
<td>How many hundreds does the sum have?</td>
</tr>
<tr>
<td><img src="image9" alt="Concrete" /></td>
<td><img src="image10" alt="Representational" /></td>
<td>1</td>
<td>“3 hundreds.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>What is the sum?</td>
</tr>
<tr>
<td><img src="image11" alt="Concrete" /></td>
<td><img src="image12" alt="Representational" /></td>
<td>300+80+2</td>
<td>“3 hundreds, 8 tens, 2 ones.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>382</td>
<td>“Three hundred eighty two.”</td>
</tr>
</tbody>
</table>
Concrete – Representation – Abstract

\[
\begin{array}{c}
1 \\
5 \ 2 \ 5 \\
+ \ 1 \ 3 \ 6 \\
\hline
6 \ 6 \ 1
\end{array}
\]

The Doubting Teacher

Do they “see” what I “see”? How do I know?
Concrete/Representational Modeling
Partner Practice

45 + 26 = 71
37 + 78 = 115
68 + 18 = 86
97 + 33 = 130

387 + 69 = 456
238 + 195 = 433
162 + 387 = 549
148 + 154 = 292
Subtraction
Partner Practice (model C/R, as teacher)

- 32 - 35 - 70 - 45
  - 4 - 26 - 13 - 17

- 202 - 350 - 294 - 230
  - 33 - 223 - 189 - 186
Multiplication
Teach the blocks!
Teach the blocks!
CRA Sequence of Instruction

1. Introduce the mathematical concept(s)
2. Teach and practice modeling procedures concretely
3. Connect the concrete to a representation of the concrete
4. Practice modeling the procedure representationally
5. Connect the representation to the abstract symbols
6. Practice the abstract modeling of the procedure
7. Make connections between all three models to help students monitor their thinking and choice of representation
8. Provide opportunities for student choice.
### Multiplication (place value & partial products)

<table>
<thead>
<tr>
<th>Abstract</th>
<th>12 = 10 + 2</th>
<th>12</th>
<th>10 + 2</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 = 20 + 4</td>
<td>24</td>
<td>20 + 4</td>
<td>How can the second factor be written in expanded form?</td>
<td></td>
</tr>
</tbody>
</table>

![Diagram](chart.png)

<table>
<thead>
<tr>
<th>Abstract</th>
<th>12</th>
<th>10 + 2</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 x 24</td>
<td>48</td>
<td>40 + 8</td>
<td>What is the product of the each number (digit) in the first factor by each number (digit) in the second factor</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>&quot;8 ones&quot;</th>
<th>12</th>
<th>10 + 2</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>no 24 x 4</td>
<td>48</td>
<td>40 + 8</td>
<td>Can I regroup 10 ones to make a ten?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>&quot;8 tens&quot;</th>
<th>12</th>
<th>10 + 2</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>no 24 x 20</td>
<td>48</td>
<td>40 + 8</td>
<td>Can I regroup 10 tens to make a hundred?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>&quot;2 hundreds&quot;</th>
<th>12</th>
<th>10 + 2</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>no 24 x 200</td>
<td>48</td>
<td>40 + 8</td>
<td>Can I regroup 10 hundreds to make a thousand?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>288</th>
<th>288</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>288</td>
<td>288</td>
<td>What is the final product?</td>
</tr>
</tbody>
</table>
Arrays & Area Models

$3 \times 2$

$A = l \times w$

$A = 2 \times 3$

$A = 6$
Concrete – Representation – Abstract

\[
\begin{array}{c c c c c}
12 & \times & 11 \\
12 \\
+120 \\
132
\end{array}
\]

\[
\begin{array}{c c c}
10 & 2 \\
10 & 100 & 20 \\
1 & 10 & 2
\end{array}
\]

= 120

= 12
Concrete/Representational Modeling
Partner Practice (model C/R, as teacher)

21 \times 4

22 \times 13

24 \times 20

17 \times 11

23 \times 12

12 \times 14

13 \times 32

11 \times 31
Connecting Models to the Abstract
Partner Practice (model C/R to A)

26 \times 5

40 \times 32

21 \times 11

10 \times 30

23 \times 12

31 \times 16

14 \times 22

39 \times 47
Division
Early Division
### Division (Place Value & Partial Products)

<table>
<thead>
<tr>
<th>Concrete</th>
<th>Representational</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Concrete Image" /></td>
<td><img src="image2" alt="Representational Image" /></td>
<td>What is the number we are dividing? or What is the dividend?</td>
</tr>
<tr>
<td><img src="image3" alt="Concrete Image" /></td>
<td><img src="image4" alt="Representational Image" /></td>
<td>What are we dividing into? or What is the divisor?</td>
</tr>
<tr>
<td><img src="image5" alt="Concrete Image" /></td>
<td><img src="image6" alt="Representational Image" /></td>
<td>How many hundreds are in each group?</td>
</tr>
<tr>
<td><img src="image7" alt="Concrete Image" /></td>
<td><img src="image8" alt="Representational Image" /></td>
<td>How many hundreds are remaining?</td>
</tr>
<tr>
<td><img src="image9" alt="Concrete Image" /></td>
<td><img src="image10" alt="Representational Image" /></td>
<td>How many tens is this worth?</td>
</tr>
<tr>
<td><img src="image11" alt="Concrete Image" /></td>
<td><img src="image12" alt="Representational Image" /></td>
<td>How many tens are in each group?</td>
</tr>
<tr>
<td><img src="image13" alt="Concrete Image" /></td>
<td><img src="image14" alt="Representational Image" /></td>
<td>How many tens are remaining?</td>
</tr>
<tr>
<td><img src="image15" alt="Concrete Image" /></td>
<td><img src="image16" alt="Representational Image" /></td>
<td>How many ones is this worth?</td>
</tr>
<tr>
<td><img src="image17" alt="Concrete Image" /></td>
<td><img src="image18" alt="Representational Image" /></td>
<td>How many ones are in each group?</td>
</tr>
<tr>
<td><img src="image19" alt="Concrete Image" /></td>
<td><img src="image20" alt="Representational Image" /></td>
<td>How many ones are remaining?</td>
</tr>
<tr>
<td><img src="image21" alt="Concrete Image" /></td>
<td><img src="image22" alt="Representational Image" /></td>
<td>What is the quotient? or How many are in one group with how many remaining?</td>
</tr>
</tbody>
</table>

### Division (Abstract Algorithms)

<table>
<thead>
<tr>
<th></th>
<th>Abstract (explicit) algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

### Division (Abstract Standard Algorithm)

<table>
<thead>
<tr>
<th></th>
<th>Abstract (standard algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Concrete – Representation – Abstract

```
3 4 2 7
-3 12 -6
1 -12 0

or 1 4 2 \frac{1}{3}
```
Why do we “bring down” the digits?

Partial quotients give partial remainders.
Partner Practice (model C/R, as teacher)

\[
\begin{align*}
3 & \div 62 \\
5 & \div 68 \\
4 & \div 71 \\
2 & \div 410 \\
4 & \div 202 \\
3 & \div 300 \\
\end{align*}
\]
Fractions
## Fractions as “labels”

<table>
<thead>
<tr>
<th>2 + 3</th>
<th>2 apples + 3 apples</th>
<th>2 fourths + 3 fourths</th>
<th>( \frac{2}{4} + \frac{3}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5 apples</td>
<td>5 fourths</td>
<td>( \frac{5}{4} )</td>
</tr>
</tbody>
</table>
Fractions as Labels

\[
\frac{4}{6}
\]
Unit Fractions

\[ \frac{a}{b} = a \times \frac{1}{b} \quad \text{counting} \quad \frac{1}{b} 's ' \]
Unit Fractions

\[
\frac{a}{b} = a \times \frac{1}{b}
\]

\[
\frac{2}{3} \text{ cup flour} = 2 \times \frac{1}{3} \text{ cup flour}
\]
Unit Fractions

\[
\frac{a}{b} = a \times \frac{1}{b}
\]

\[
\frac{1}{2} = \frac{2}{4} = \frac{4}{8}
\]
Educational Consultants

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Commonwealth of Pennsylvania
Tom Wolf, Governor

PaPiNAN