NAC 2018

#ptnmath

Mathematics in Autism Interventions

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PaTAN on twitter

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PaTTAN's Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.

PDE's Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.

You are the expert with your students!

Session Outline

Quantitative verbal concepts begin with the ability to tact quantities and extends across exemplars. This session will introduce participants to fundamental verbal concepts in mathematics. Participants will be able utilize a skills sequence to teach these concepts.

Objectives

- Participants will extend their understanding of teaching students to Tact items to build students' conceptual knowledge
- Participants will be able determine apply quantitative concepts across mathematical domains
- Participants will be able to identify multiple exemplars for mathematical concepts
- Participants will understand the progression of skill sequence when using the CRA model of instruction

Session Outline

1. Application of ABA and Verbal Behavior to mathematic Instruction
2. Concept of Number
3. Two-way Quantitative Verbal Concepts
4. One-way Quantitative Verbal Concepts

“But counting is so simple, I know he can learn it”

Teaching each symbol or Teaching the collection

Each Symbol

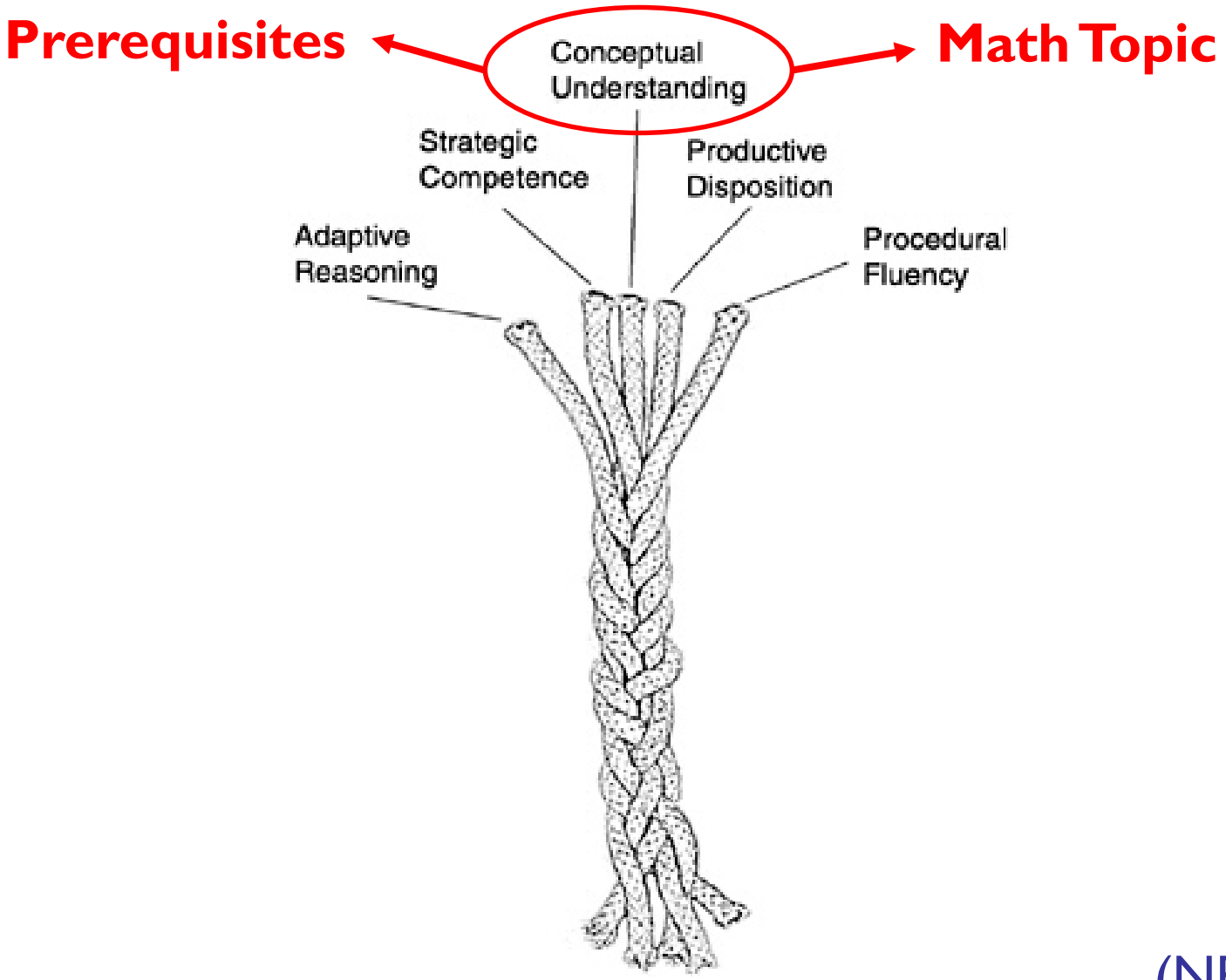
- Name – Meaning – Quantity
- Ability to Subitize

Collection

- Counting
- Magnitude
- Applications

Across Exemplars

5 Strands of Mathematical Proficiency



(NRC, 2001)

What is **conceptual understanding?**

Extended Tacts

- Generalization must occur
  - Can apply to novel items without explicit teaching
  - Across…
    1. People
    2. Places
    3. Materials
    4. Instructions
    5. Time
- Feature/Function/Class
  - Tacting critical features may facilitate concept acquisition
- The tact is involved in the process of joint control which assists students in effective verbal recall and effective listener responding

What is **conceptual understanding?**

Atomic Repertoires

- New combination of skills applied to new behaviors
- Most of our spoken language is a result of ARs

What are the prerequisite skills needed for the atomic repertoires for the math content?

- Imitation
- Echoic
- Tacts
- Textual Behavior (reading texts/symbols)
- Transcriptive Behavior (copying text/symbols)
- Etc…

We must identify and teach the skills!

Textual Behavior…

It is important for students to be able to "read" mathematics.

However, textual behavior is only relevant when students understand the meaning of the words.

OR

Interpreting math symbols is only relevant when they understand their meaning.

Quantitative Verbal Concepts

"Language used to describe something that is connected to a value (numerical/spatial)"

More later…….

Now let's talk **math!**

Concept of Number

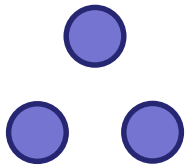
"3"

"What does three really mean?
What is three-ness"

-MM

What does “3” really mean?

3 three *“three”*



"1 … 2 … 3! "

“one more than 2”

“one less than 4”

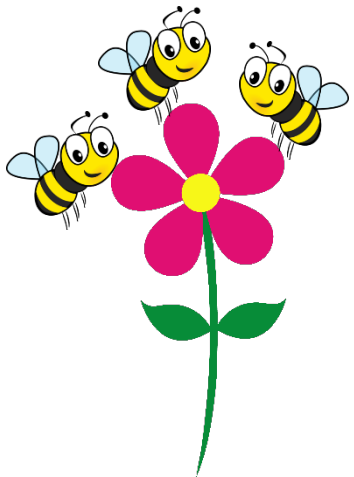
“is between… ”

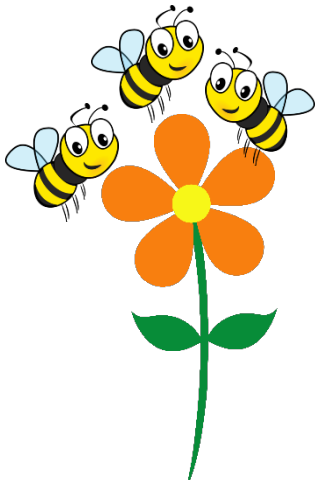
“is more than… ”

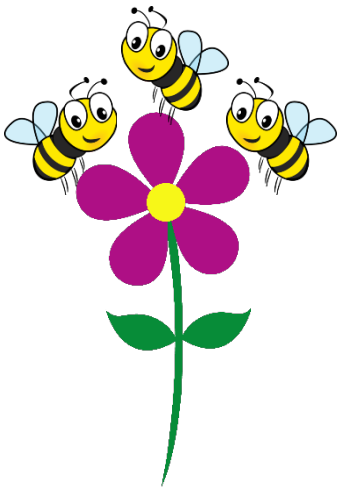
“is less than… ”

“is the same as… ”

3 *units*







What is Number Sense?

"a child's fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons"

(Gersten & Chard, 1999)

"Concept Matrix"

Student (behavior)

Teacher (antecedent)

	Find digit	Write digit	Write text	Say number	Make/select pattern
Say number	LR	Trans.	Trans.	Echoic	LR
Show digit	MtS	Trans.	IV	IV	MtS
Show text	MtS	IV	Trans.	Text	MtS
Show pattern	MtS	IV	IV	Tact	MtS

CAUTION!

- Before you begin math instruction:
  - Can student say (echo) necessary responses?
  - Do they already have a broad tact repertoire?
  - Are they not making conditional discrimination errors?
  - Are they mediating their response when problem solving (answering questions, following multiple component directions)?
  - Can they write necessary responses?

Symbols & Meaning

- Two ways to understand letters…
  - "B" is the letter "bee" and makes the sound */b/*
- What about numbers?
  - Names are taught
  - Meaning is based on place value (base 10)

Number Names & Meanings

#	Name	Meaning	#	Name	Meaning
0	Zero	None	20	Twenty	Two tens
1	One	One	21	Twenty-one	Two tens, one
2	Two	Two	22	Twenty-two	Two tens, two
3	Three	Three	23	Twenty-three	Two tens, three
4	Four	Four	24	Twenty-four	Two tens, four
5	Five	Five	25	Twenty-five	Two tens, five
6	Six	Six	26	Twenty-six	Two tens, six
7	Seven	Seven	27	Twenty-seven	Two tens, seven
8	Eight	Eight	28	Twenty-eight	Two tens, eight
9	Nine	Nine	29	Twenty-nine	Two tens, nine
10	Ten	One ten	30	Thirty	Three tens
11	Eleven	One ten, One	31	Thirty-one	Three tens, one
12	Twelve	One ten, Two	32	Thirty-two	Three tens, two
13	Thirteen	One ten, Three	**Other examples**		
14	Fourteen	One ten, Four	48	Forty-eight	Four tens, eight
15	Fifteen	One ten, Five	53	Fifty-three	Five tens, three
16	Sixteen	One ten, Six	62	Sixty-two	Six tens, two
17	Seventeen	One ten, Seven	75	Seventy-five	Seven tens, five
18	Eighteen	One ten, Eight	81	Eighty-one	Eights tens, one
19	Nineteen	One ten, Nine	99	Ninety-nine	Nine tens, nine

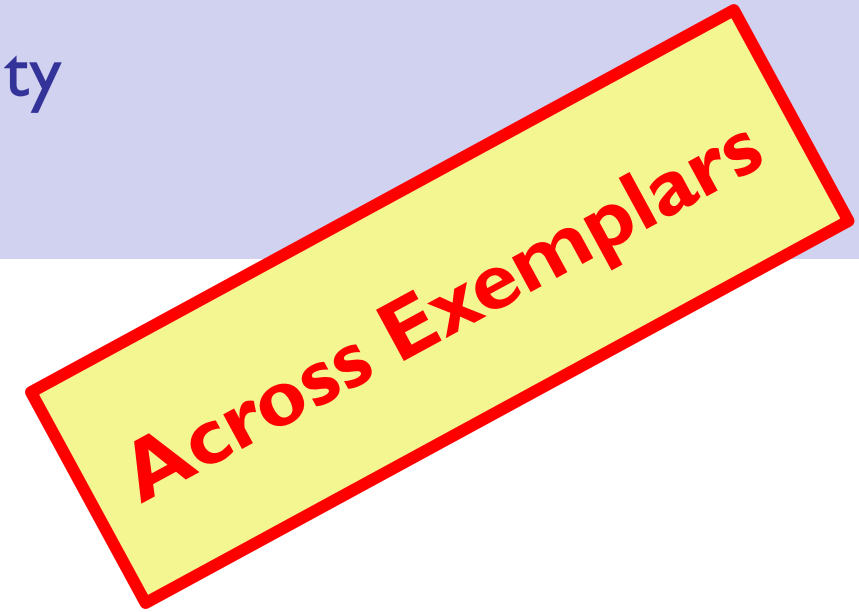
Teaching each symbol or Teaching the collection

Each Symbol

- Name – Meaning – Quantity
- Ability to Subitize

Collection

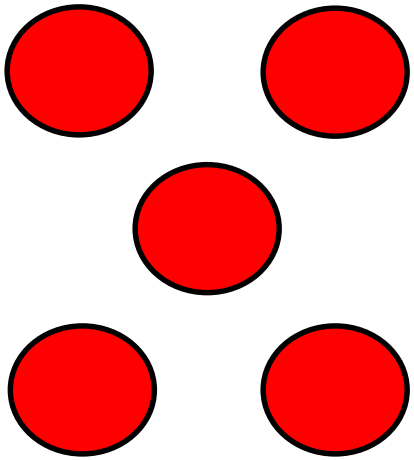
- Counting
- Magnitude
- Applications

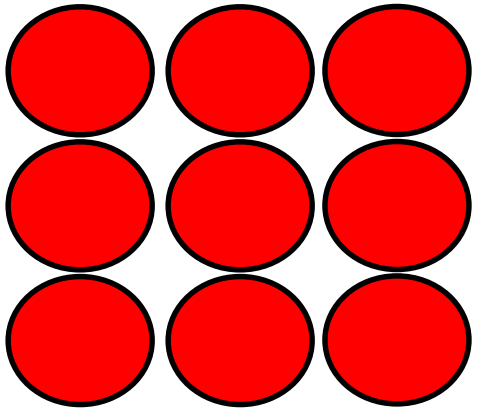


Subitize

The ability to see a quantity and know how many, without "counting."

Perceptual and **Conceptual**





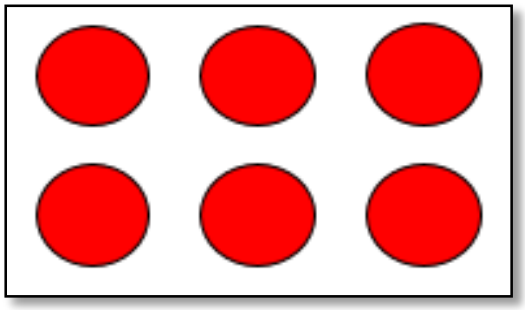
Subitization

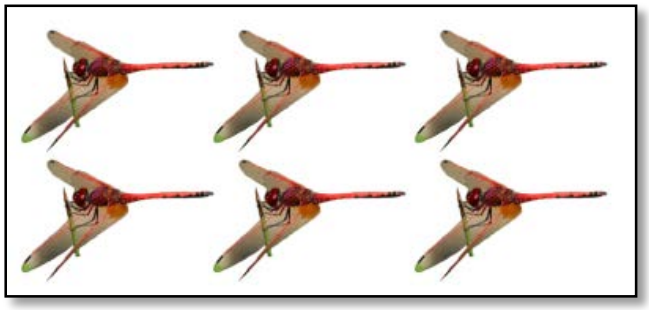
Research indicated that dice patterns and rectangular arrays are the easiest for students to learn.

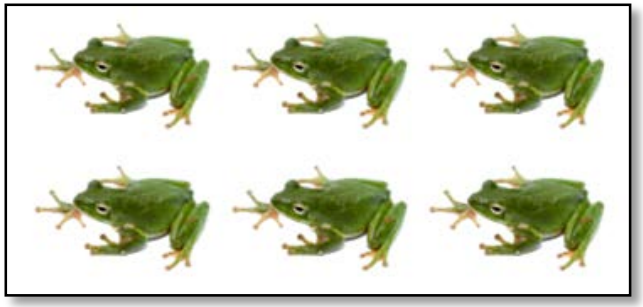
Don't go crazy!

Clements, D. H. (1999). Subitizing: What is it? Why teach it?. Teaching children mathematics, 5(7), 400.

Subitization



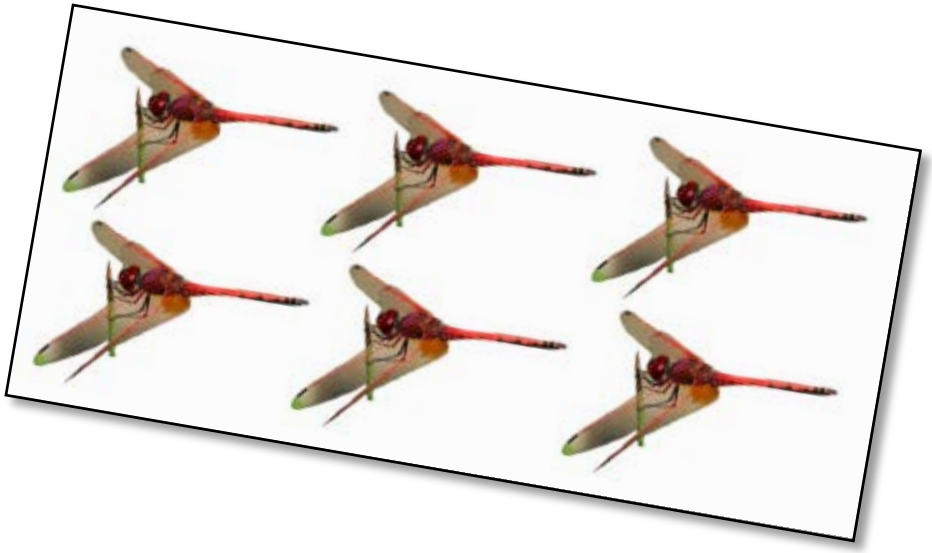




Subitization – Tacting a Feature

Verbal Conditional Discrimination must be established.

- What is it?
- What part is it?
- How many?



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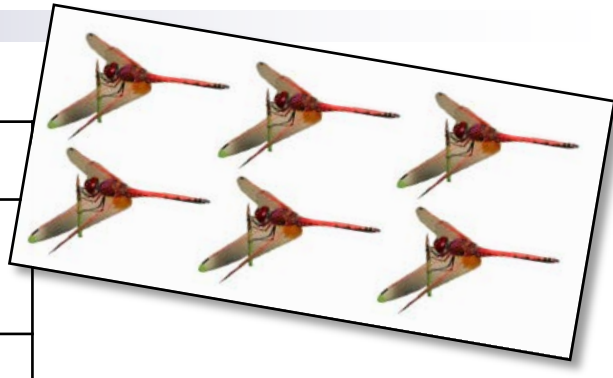
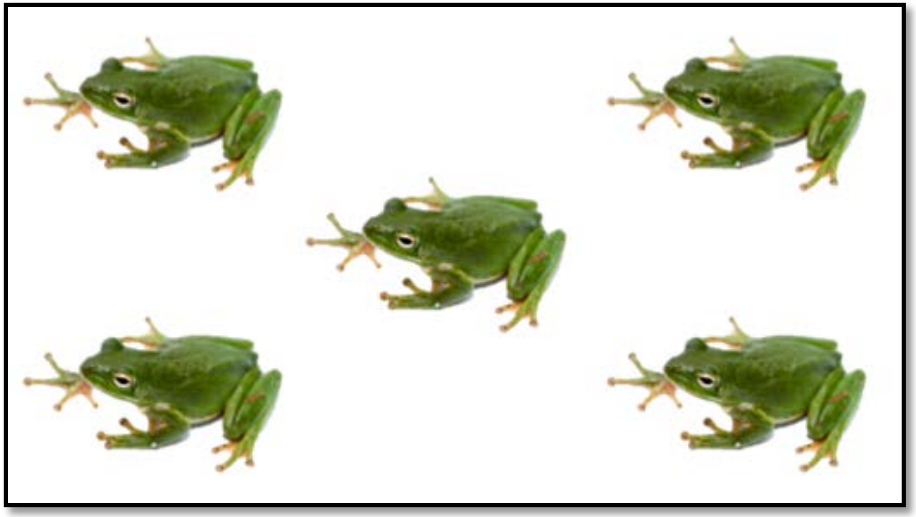


Trial	Teacher	Learner
Tact Prompt for Feature	Presents item "How many? Six."	"Six"
Tact Transfer	"How many?"	"Six"
Distractor(s)	?	?
Tact Trial Item	Presents item "What are these?"	"Red-veined Dropwing Dragonflies"
Tact Feature Check	Presents item "How many?"	"Six"

Error Correction – Run a contrast correction as part of the distract trial sequence

Subitization – Tacting a Feature



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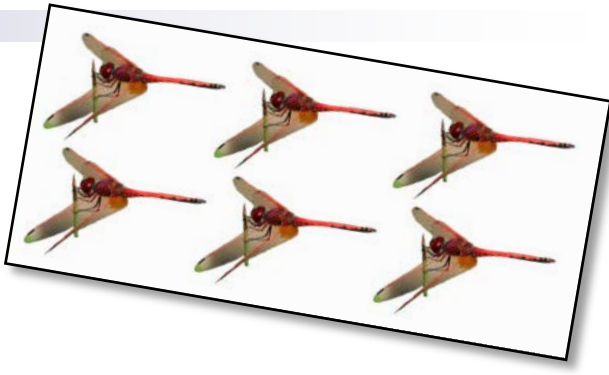
Subitization – Tacting a Feature



Generalization & discrimination should be present for the items in the set.

The concept of quantity has been developed when the individual can subitize (tact) the number of novel items in a set without explicit training.

What is CRA?

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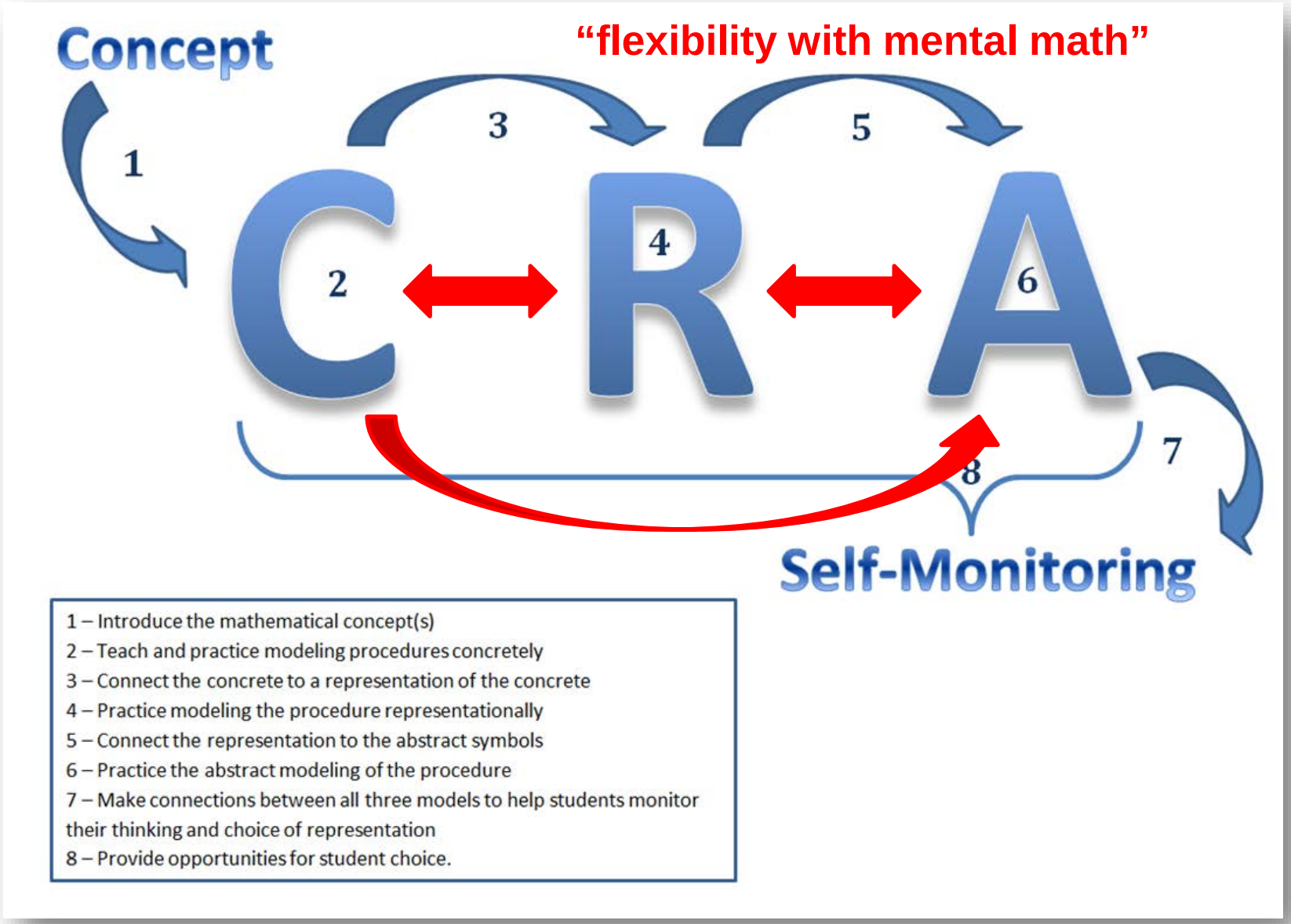
CRA Sequence of Instruction



CRA

- Concrete (sense making by moving)
- Representational (sense making by drawing)
- Abstract (sense making with symbols)

CONSISTENT LANGUAGE

Rationale – Doing What Works

Research-based studies show that students who use concrete materials develop **more precise and more comprehensive mental representations**, often show more motivation and on-task behavior, understand mathematical ideas, and better apply these ideas to life situations.

(Harrison, & Harrison, 1986)

(Suydam & Higgins, 1977)

Why would CRA be effective?

- Multiple responses for each mathematical concept to aid "memory and retrieval"
- Meaningful manipulations of materials allows students to engage in overt responses to assist in problem solving abstract mathematics
- Procedural accuracy; provides alternate to algorithm memorization

(Witzel, Riccomini, & Scheider, 2008)

Other Research.

- Direct Instruction
- Errorless Teaching
- Formative Assessment
- Correct Feedback
- Improved Teacher Content Knowledge
  - Task Analyze
  - Instruct on Specific Skills or Process
  - Monitor progress
  - Correct errors

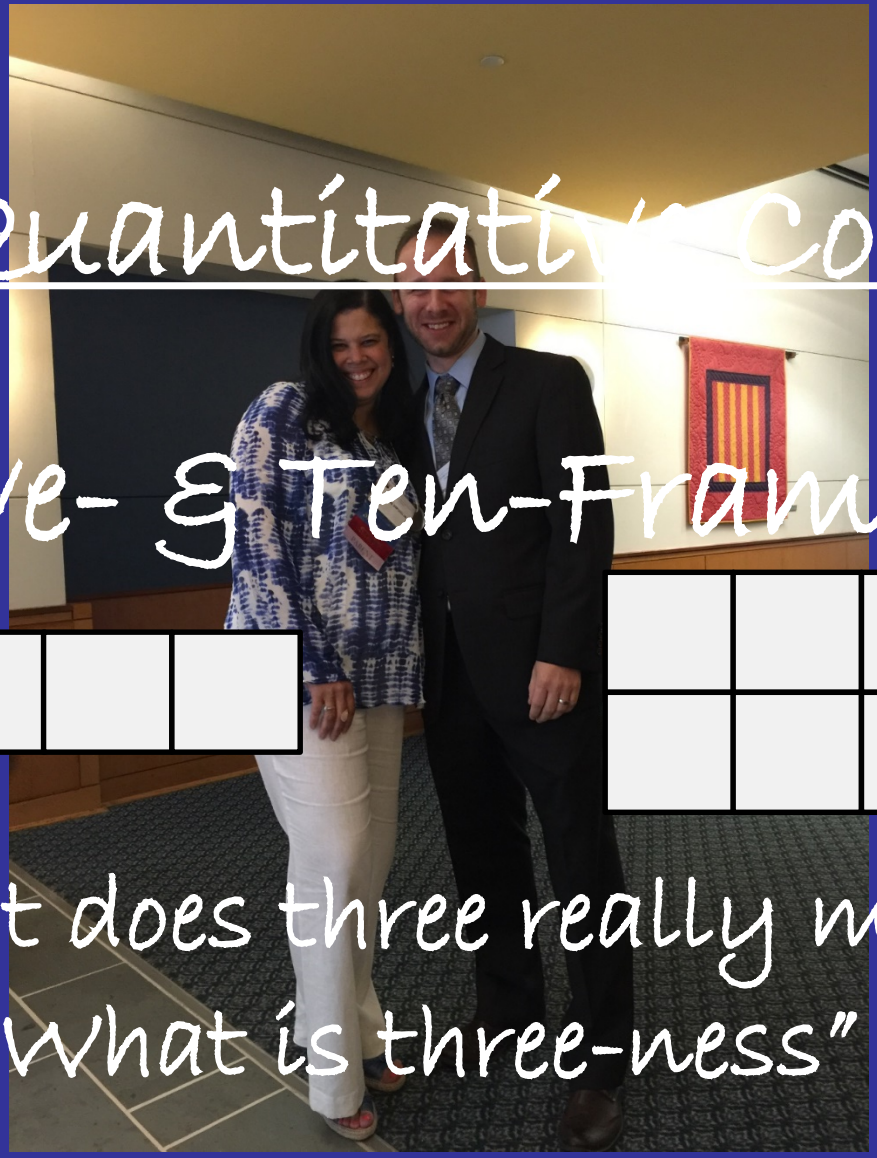
Students having difficulties with math...

- Counting seen as rote, mechanical, left to right, 1: 1 correspondence only; INEFFECTIVE
- Automaticity problems interfere with concept formation and problem solving

(Gersten, Jordan, & Flojo, 2005)

Early Quantitative Concepts:

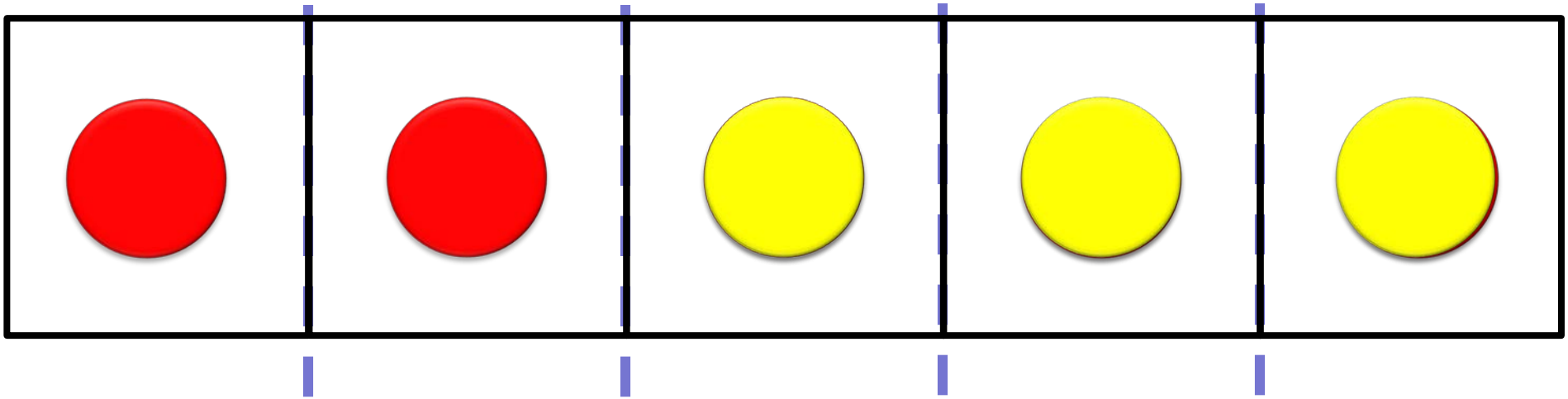
Five- & Ten-Frames



"What does three really mean?
What is three-ness"

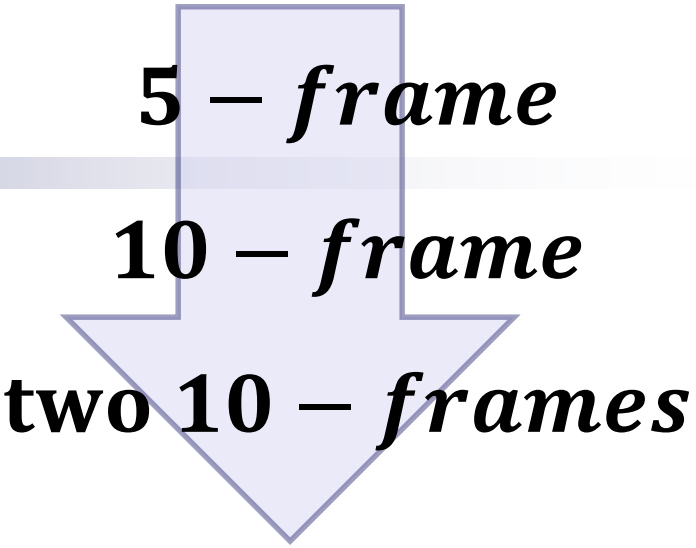
-MM

Decomposition



see the parts & see the whole

Purpose of 10-frame



- See sets of 5
- See sets of 10
- Organize in rectangular array
  - Subitize
- Reduces need to "count"
- Visually decompose numbers in sets of 5

Subitization – Ten Frames

Subitization – Ten Frames

Subitization – Ten Frames

Five Counting Principles

3 **One-to-one** – Counting one "thing" at a time; transfer from uncounted group to counted group ($\mathbf{1:1}$ $\boldsymbol{Correspondance}$)

2 **Cardinal** – The last count represent the quantity in the counted group ($\boldsymbol{Cardinality}$)

1 **Stable-order** – Establishes consistent sequence

Abstraction – applying counting to like objects, actions, sounds, etc…

4/5

Order-irrelevance – Can count in any order

Early Instructional Sequence

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Suggested Sequence

- TARGET SKILLS 1-6 Tact Typical Dice Pattern
- TARGET SKILLS 7-12: LR –Select Typical Dice Pattern (will likely get for free with teaching tact)
- TARGET SKILLS 13-18: Build Dice Pattern (start as soon as possible once initial patterns acquired as tact/LR)
- TARGET SKILLS 19-23: Tact Various Atypical Dice Pattern (on card)
- TARGET SKILLS 24-28:Tact Various Atypical Dice Pattern (with objects)
- TARGET SKILLS 29-33: Tact Dice Pattern (on card) in discrimination
- TARGET SKILLS 34-38: Tact Dice Pattern (with objects) in discrimination

Suggested Sequence

- TARGET SKILLS 39-44: Copy written numbers
- TARGET SKILLS 45-50: Write upon dictation
- TARGET SKILLS 51-56: Tact Numeral
- TARGET SKILLS 57-62: LR Select Numeral
- TARGET SKILLS 63-68: Tact Solid Pattern on 10 frame with objects
- TARGET SKILLS 69-73: Tact Solid Pattern on 10 frame with objects in discrimination
- TARGET SKILLS 74-79: Tact Solid Pattern on 10 frame (on card)
- TARGET SKILLS 80-84: Tact Pattern on 10 frame (on card) in discrimination

Suggested Sequence

- TARGET SKILLS 85-89: Build 10 frame with objects (1-5)
- TARGET SKILLS 90-93: Tact Quantity of Each Item & Altogether on Dice Pattern
- TARGET SKILLS 94-97: Tact Quantity of Each Item & Altogether on 10 frame Pattern
- TARGET SKILLS 98-102: Draw 10 frame (1-5)
- TARGET SKILLS 103-108: Tact and Select symbols (+ - =)
- TARGET SKILLS 109-113: Conceptual addition with 10 frames (up to 5)
- TARGET SKILLS 114-118: Tact Solid Pattern on 10 frame with objects (6-10)
- TARGET SKILLS 119-123: Tact Solid Pattern on 10 frame (on card) (6-10)

Suggested Sequence

- TARGET SKILLS 124-128: Tact Solid Pattern on 10 frame on card (6-10)
- TARGET SKILLS 129-133: Tact Pattern on 10 frame (on card) in discrimination (6-9)
- TARGET SKILLS 134-138: Build 10 frame with objects (6-10)
- TARGET SKILLS 139-143: Draw 10 frame (6-10)
- TARGET SKILLS 144-148: Tact Numeral
- TARGET SKILLS 149-153: LR Select Numeral
- TARGET SKILLS 154-158: Copy Written Word
- TARGET SKILLS 159-163: Write Upon Dictation
- TARGET SKILLS 164-173: Count to ____)

Suggested Sequence

- TARGET SKILLS 174-184: Read number Word
- TARGET SKILLS 185-195: Selects number Word
- TARGET SKILLS 196-206: Copy Written Word
- TARGET SKILLS 207-217: Write Upon Dictation
- TARGET SKILLS 218-227: 1:1 Correspondence Count to:
- TARGET SKILLS 228-236: Give Specified Number of Objects from Larger Set

Understanding Individual Quantities

Responding fluently to a wide range of exemplars (materials, arrangement, etc…) of quantity across operants

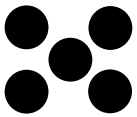
- ✓ Prerequisites
  - Level 3 students on VBMAPP, with strong verbal conditional discrimination
  - Ensure/strengthen Echoic/Imitation

Quantitative Patterns

1. Solid Dice Patterns (0-5)
   - Select and build solid patterns
2. Dice Patterns in Discrimination (0-5 + other)
3. Dice Patterns in Discrimination (0-5, within)
4. Solid Ten Frame Patterns (0-5)
   - Select and build solid patterns
5. Ten Frame Patterns in Discrimination (0-5)
6. Ten Frame Patterns in Discrimination (0-5 + others)

Indentify black

(1)



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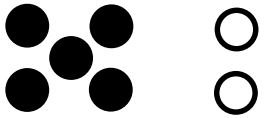

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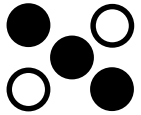

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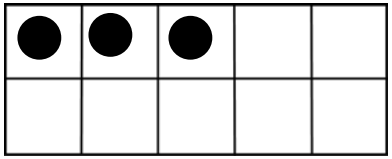

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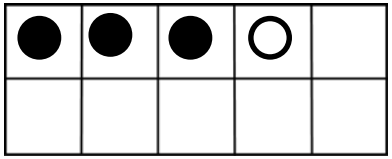

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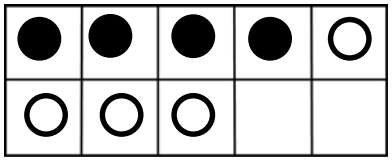
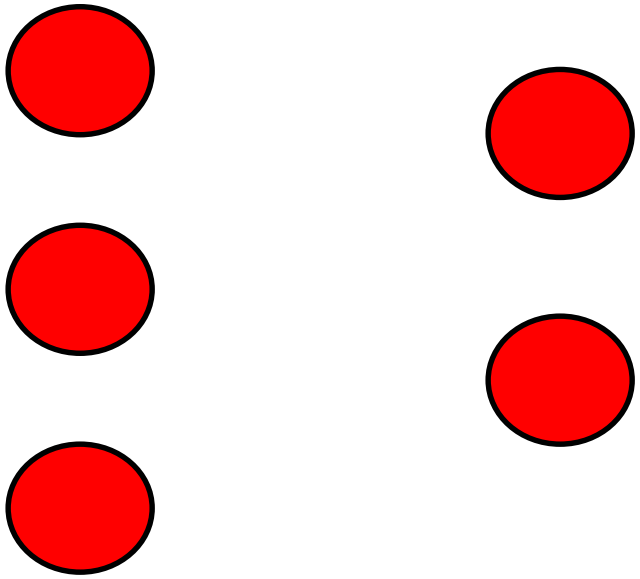


Subitizing & Conceptual Counting/Addition

$$3 + 2 = 5$$



Subitizing & Conceptual Counting/Addition

Responding fluently to a wide range of exemplars (materials, arrangement, etc…) of quantity across operants

- ✓ Prerequisites
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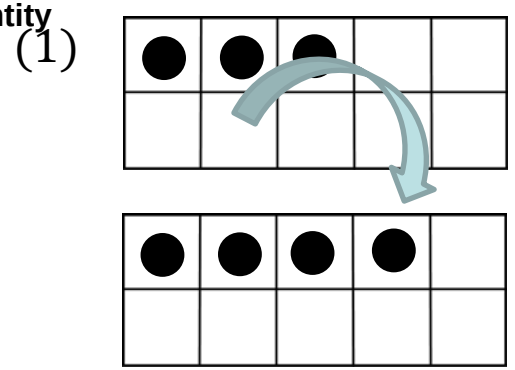
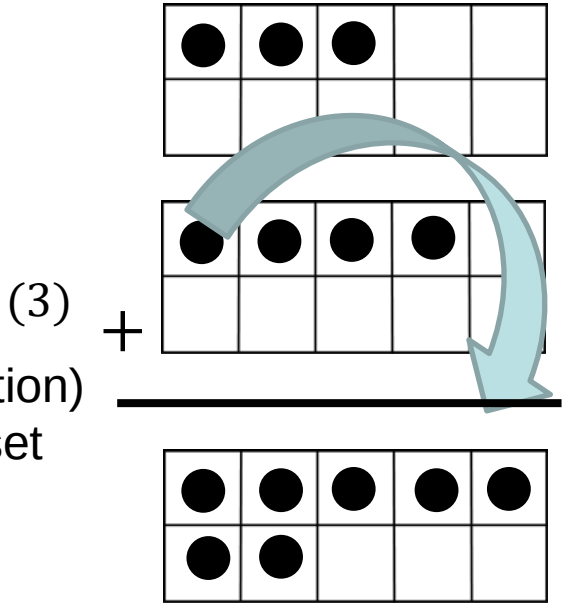
  - Ensure/strengthen Echoic/Imitation



Sets/Subsets and Digits ("Conceptual Addition")

1. Addition (+) as "put together"
2. Equal (=) as "same value"
3. Use symbols with manipulatives (horz+vert)
   - In place and as model
   - One color per problem, but vary
   - Two colors
   - Random colors
   - Other objects, same then random (i.e. generalization)
4. Interpret digits to build/select subsets and determine set

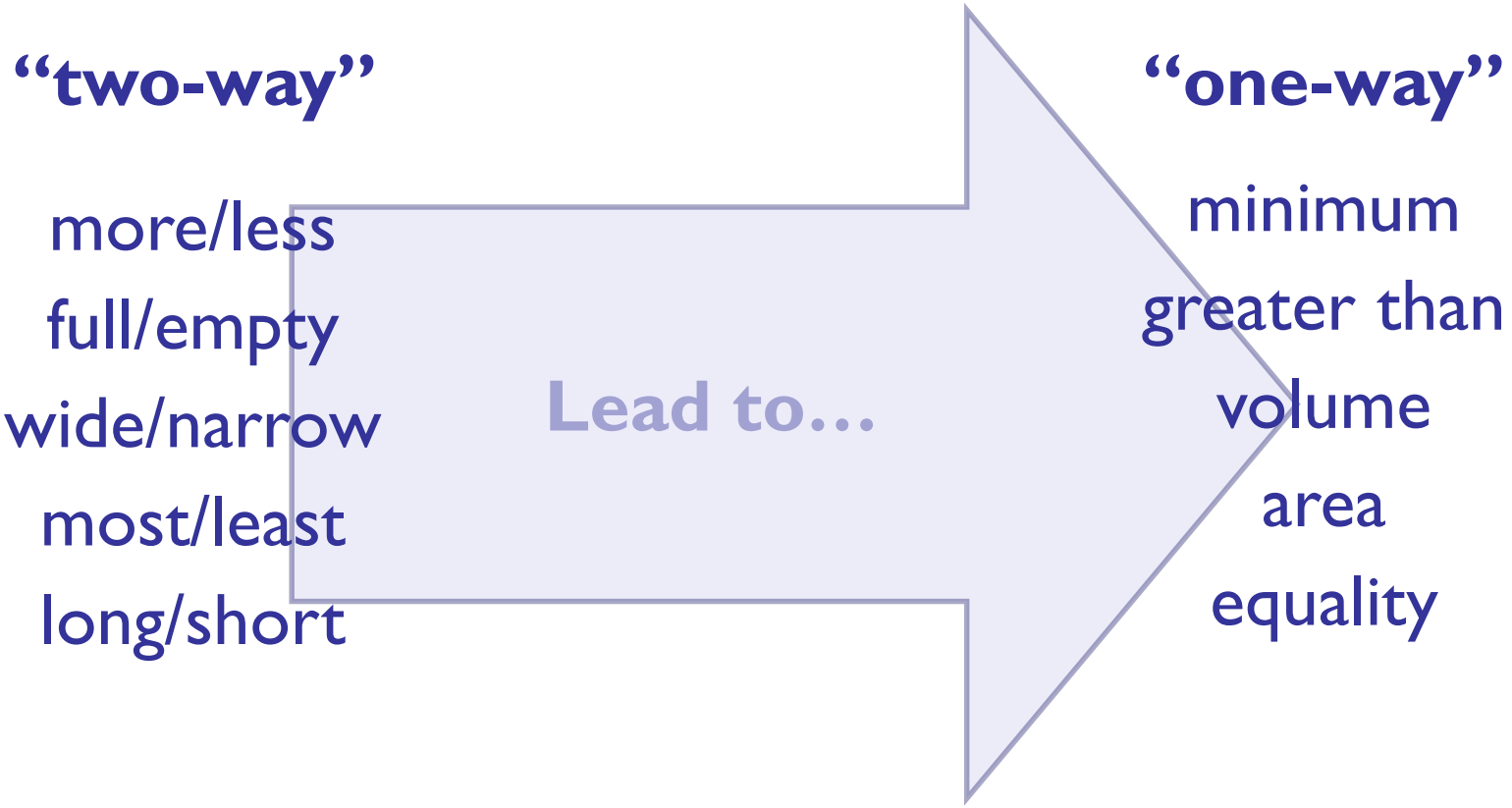


Quantitative Verbal Concepts

Pennsylvania Training and Technical Assistance Network

Quantitative Verbal Concepts (QVC)

"Language used to describe something that is connected to a value (numerical/spatial)"



Two-way QVCs

"Language used to describe something that is connected to a value (numerical/spatial)"

more/less

full/empty

wide/narrow

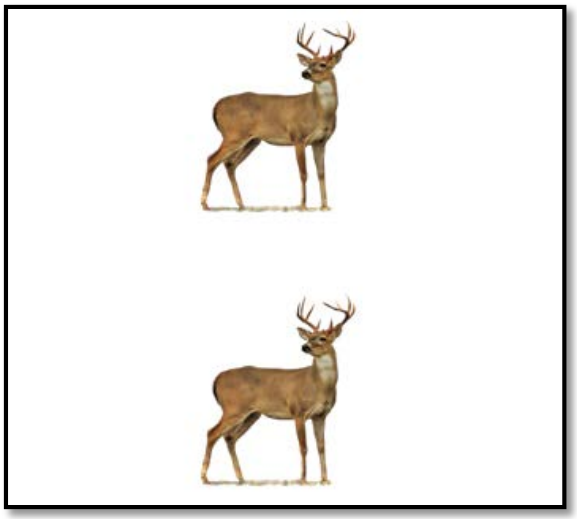
most/least

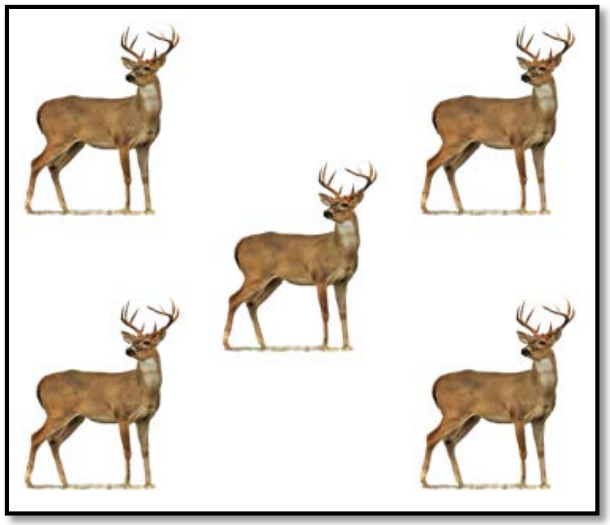
long/short

Two-way QVC – Tacting Adjectives

Trial	Teacher	Learner
Tact Trial with echoic prompt	Presents dice patterns of 5 and 2, identical except for attribute. "Let's talk about more and less." Point to 5. "This one is… *more*."	"More"
Tact Transfer	"This one is…"	"More"
Tact Trial with echoic prompt	Point to 2. "This one is… *less*."	"Less"
Tact Transfer	"This one is…"	"Less"
Distractor(s)	?	?
Tact Check	Point to 2. "This one is…"	"Less"
Distractor(s)	? (include "How many?")	?
Tact Check	Point to 5. "This one is…"	"More"
Continue with tact checks. Vary tact checks with distractors as shown above. End with the below tact check.		
Tact Check	"What are these?"	"Planes"

Two-way QVC – Tacting Adjectives





Trial
Tact Trial with echoic prompt
Tact Transfer
Tact Trial with echoic prompt
Tact Transfer
Distractor(s)
Tact Check
Distractor(s)
Tact Check
Continue with tact checks. Vary tact checks with distractors as shown above. End with the below tact check.
Tact Check item

Two-way QVC – Data Collection

	Date introduced	Date Mastered
Target: more/less		
Identical Sets:		
Single Digits		
Dice Patterns		
Ten Frames		
2 Novel Identical		
Planes		
Trucks		
Mixed Mastered pairs		
Known with one novel		
NET		
Intraverbal Opposites:		

One-way QVCs

"Language used to describe something that is connected to a value (numerical/spatial)"

minimum

greater than

volume

area

equality

fraction

What are the 2-way QVCs (adjectives) related to this?

What are the other prerequisites?

How is this concept generalized?

Equality

Algebra!

"A limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation."

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

"Students who understand the equal sign as a relational symbol of equivalence are more successful solving algebraic equations than their peers who do not have such an understanding."

Knuth, E.J., Stephens, A.D., McNeil, N.M., and Alibali, M.W. (2006) Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37, 297–312.

Equality Misconception(s)

- Children see = as a "do this" command or "complete the operations"
- Children should be able to also see = as "is the same as"

$2 + 3 = ?$

$5 - 3 = ?$

$5 - 2 = ?$

Do the operation…

$9 + 3 = \boxed{12} + 6$

$9 + 3 = \boxed{12} + 6 = 18$

$9 + 3 = \boxed{18} + 6$

"9 plus 3 is 12 ; plus 6."

"9 plus 3 is the same as 12 plus 6."

"9 plus 3 is the same as 6 plus 6."

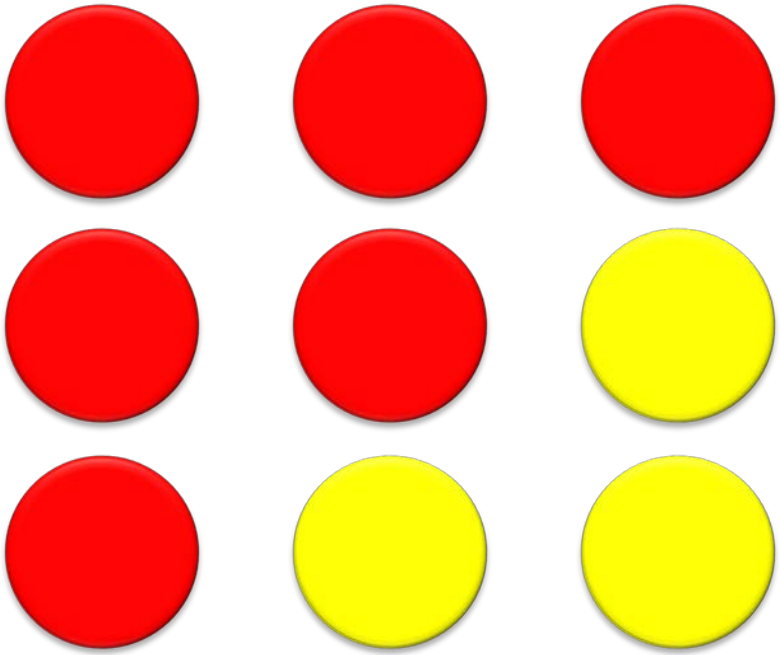
$9 + 3 = \boxed{6} + 6$

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Falkner, K.P., Levi, L. and Carpenter, T.P. (1999) Children's Understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, 6, 232–236.

Equality

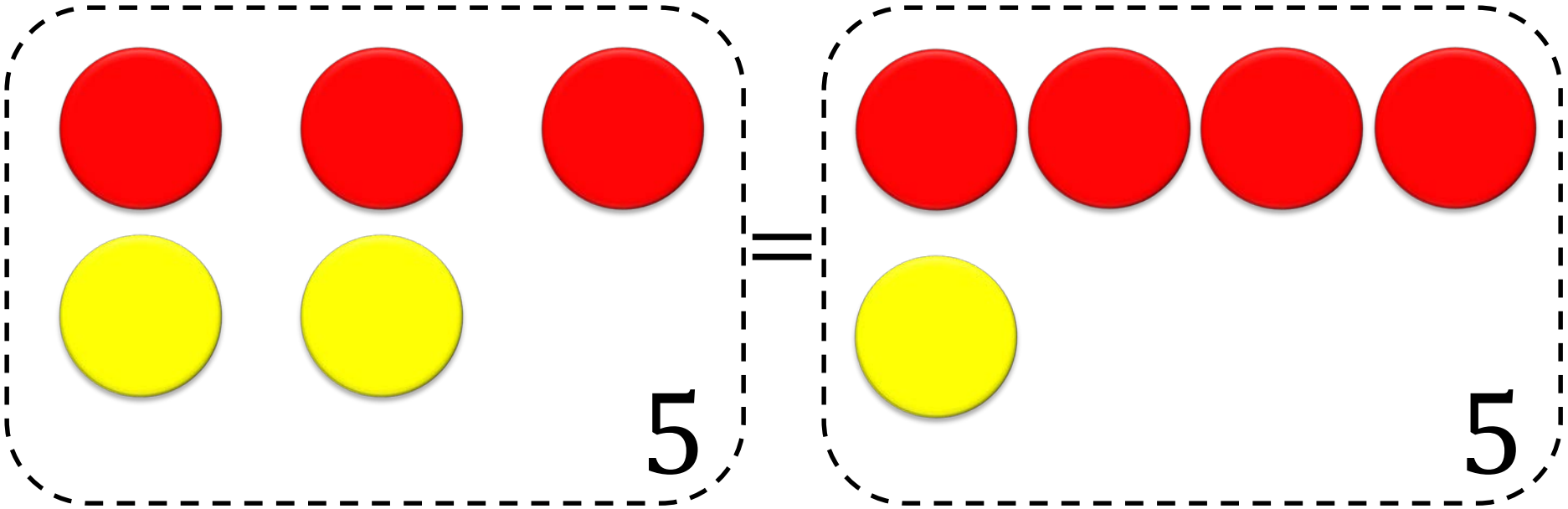
Algebra!



$$3 = 2 + 1 = 1 + 2$$

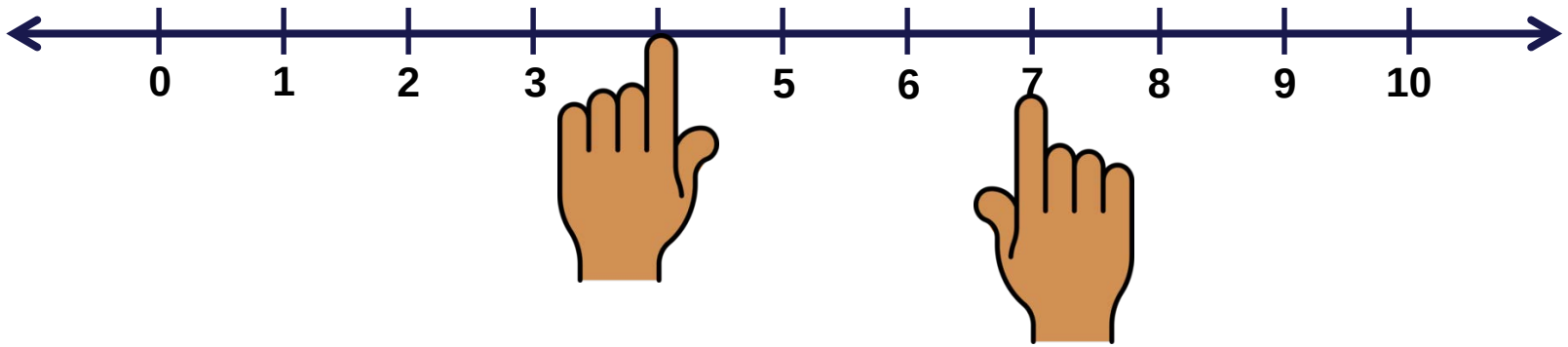
Equality

Algebra!



$3 + 2 = \boxed{4} + 1$

Magnitude on Number Line



Touch 4.

Extend more/less.

"Which side is more?"

"Touch 7. Is 7 more or less than 4?"

$x > 4$

"Which side is greater than (less than) 4 (#)?"

"Is 7 (#) greater than (less than) 4(#)?"

$x < 4$

Place Value

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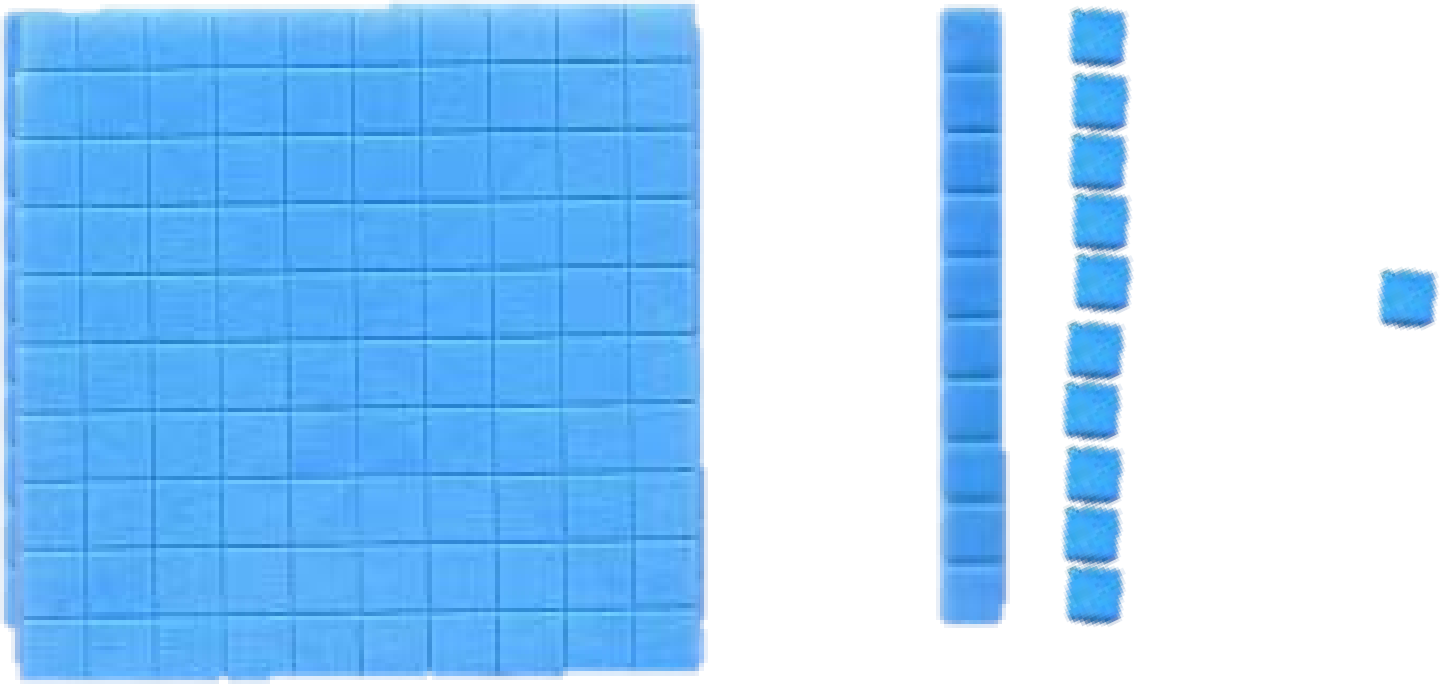
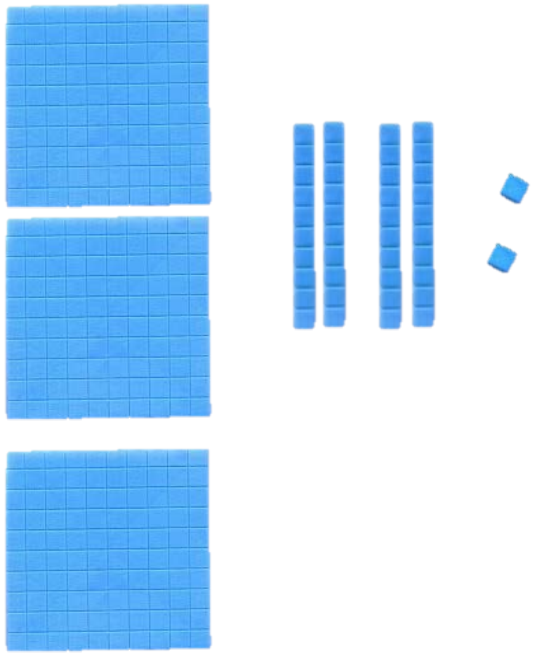
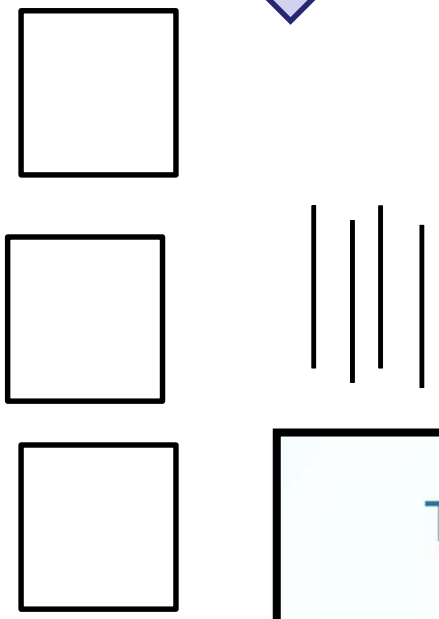
Teach the blocks!



Concrete – Representation – Abstract





342

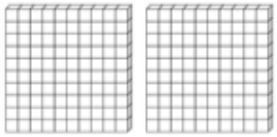
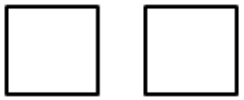
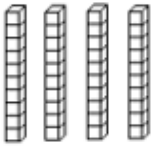
$300+40+2$

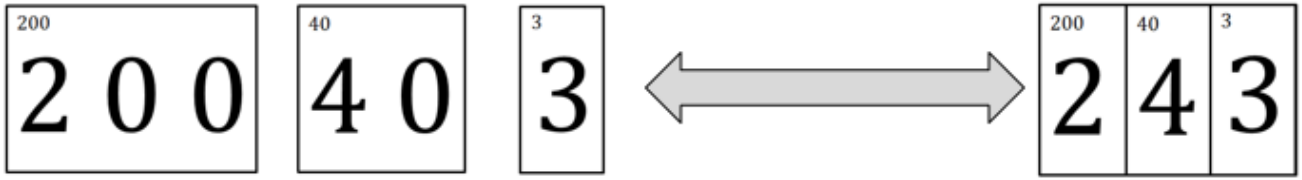
The Doubting Teacher

Do they “see” what I “see”?
How do I know?

Available online…

Place Value (expanded form)

Concrete	Representational	Abstract	Language
		243; 2 *hundreds* 200	How many hundreds? What is the value?
		243; 4 *tens* 40	How many tens? What is the value?
		243; 3 *ones* 3	How many ones? What is the value?
Reread the problem.			
$200 + 40 + 3$			How do we write the number in expanded form?
"two hundred forty three"			How do we read the number?
May support the Abstract portions with place value cards			



Addition

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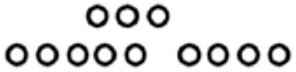
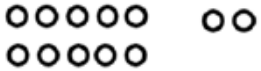
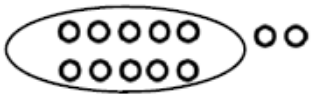
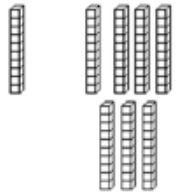
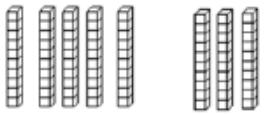
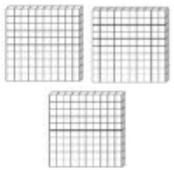
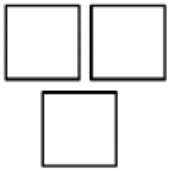
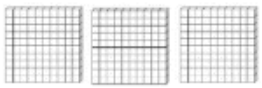
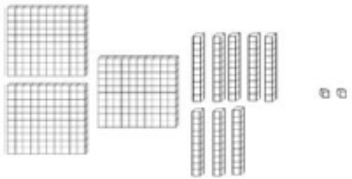
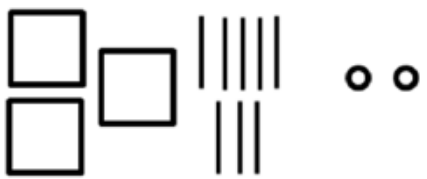
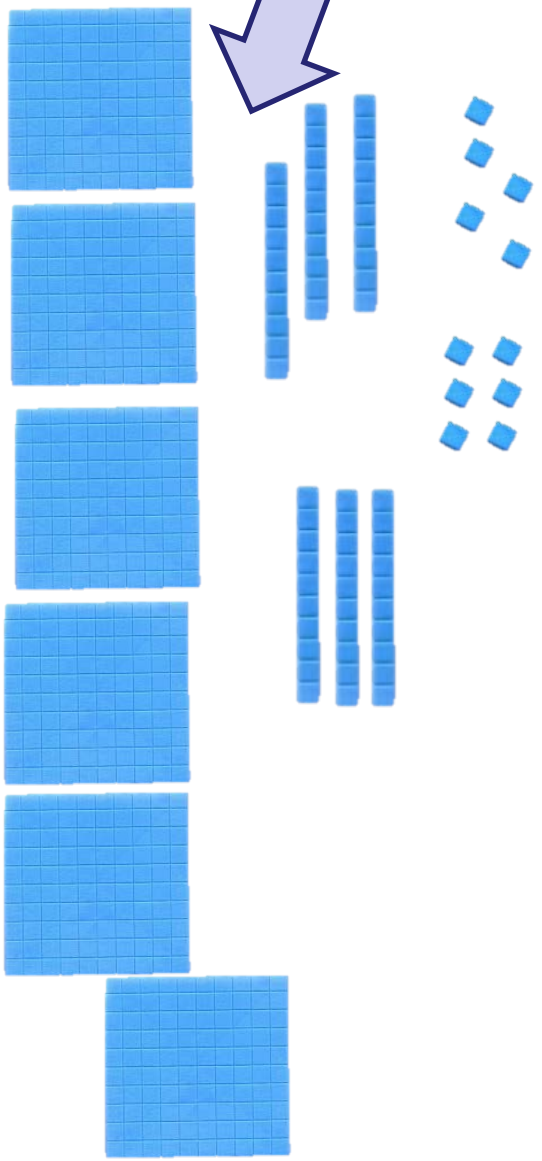
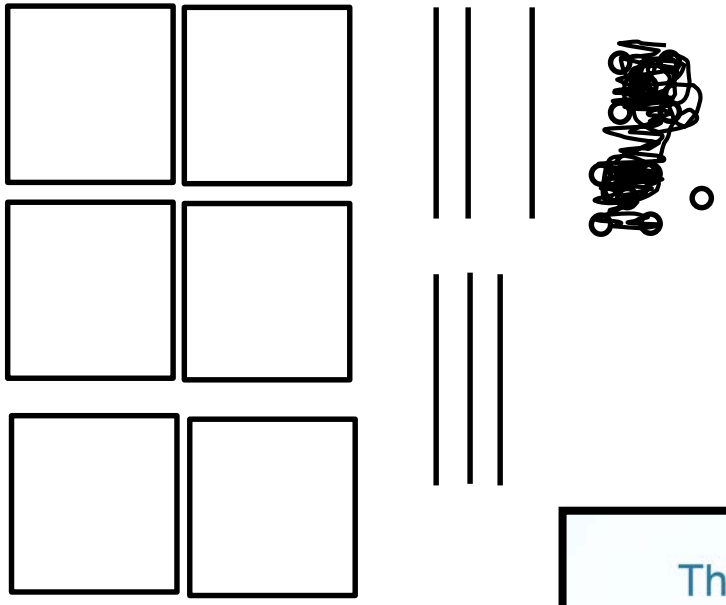
Addition (right-left)

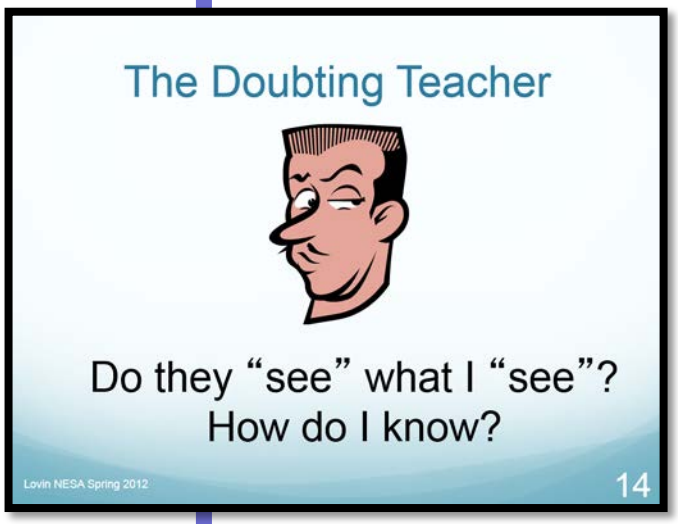
Concrete	Representational	Abstract	Language
		243 +139	How many ones does each number have? *"3 ones and 9 ones"*
		1 243 +139 2	How many ones does the sum have? *"12 ones."*
			Can I regroup 10 ones to make a ten? *"Yes, 1 ten and 2 ones."*
		1 243 +139 82	How many tens does each number have? *"1 ten, 4 tens, and 3 tens."*
			How many tens does the sum have? *"8 tens."*
No	**No**		Can I regroup 10 tens to make a hundred? *"No."*
		1 243 +139 382	How many hundreds does each number have? *"2 hundreds and 1 hundred."*
			How many hundreds does the sum have? *"3 hundreds."*
		300 + 80 + 2 382	What is the sum? *"3 hundreds, 8 tens, 2 ones."* *"Three hundred eighty two."*

Concrete – Representation – Abstract





$$\begin{array}{r} {\scriptstyle 1} \\ 525 \\ +\ 136 \\ \hline 661 \end{array}$$



Concrete/Representational Modeling

Partner Practice

$$\begin{array}{r} 45 \\ +\,26 \\ \hline \end{array} \qquad \begin{array}{r} 37 \\ +\,78 \\ \hline \end{array} \qquad \begin{array}{r} 68 \\ +\,18 \\ \hline \end{array} \qquad \begin{array}{r} 97 \\ +\,33 \\ \hline \end{array}$$

$$\begin{array}{r} 387 \\ +\;\;69 \\ \hline \end{array} \qquad \begin{array}{r} 238 \\ +\,195 \\ \hline \end{array} \qquad \begin{array}{r} 162 \\ +\,387 \\ \hline \end{array} \qquad \begin{array}{r} 148 \\ +\,154 \\ \hline \end{array}$$

Subtraction

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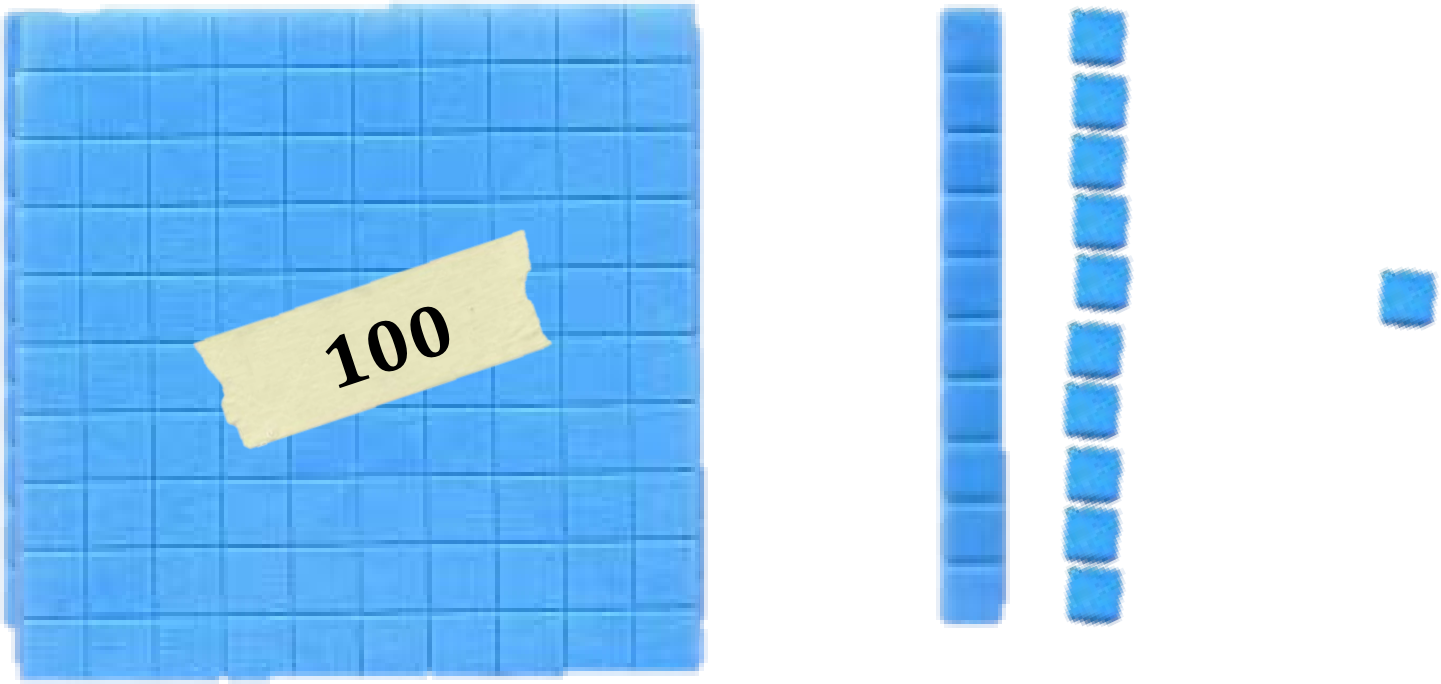
Partner Practice (model C/R, as teacher)

$$\begin{array}{r} 32 \\ -\ 4 \\ \hline \end{array} \qquad \begin{array}{r} 35 \\ -\ 26 \\ \hline \end{array} \qquad \begin{array}{r} 70 \\ -\ 13 \\ \hline \end{array} \qquad \begin{array}{r} 45 \\ -\ 17 \\ \hline \end{array}$$

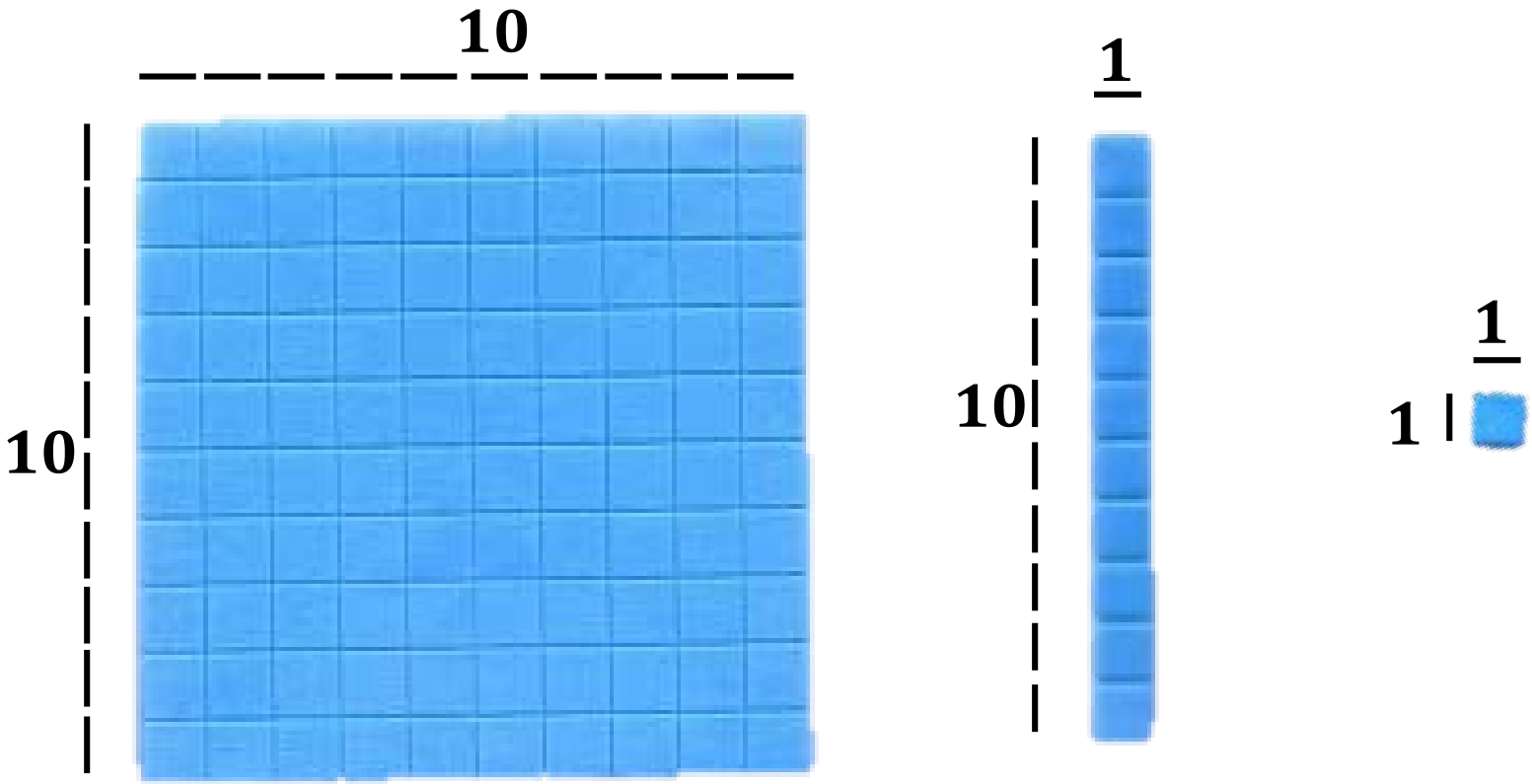
$$\begin{array}{r} 202 \\ -\ 33 \\ \hline \end{array} \qquad \begin{array}{r} 350 \\ -\ 223 \\ \hline \end{array} \qquad \begin{array}{r} 294 \\ -\ 189 \\ \hline \end{array} \qquad \begin{array}{r} 230 \\ -\ 186 \\ \hline \end{array}$$

Multiplication

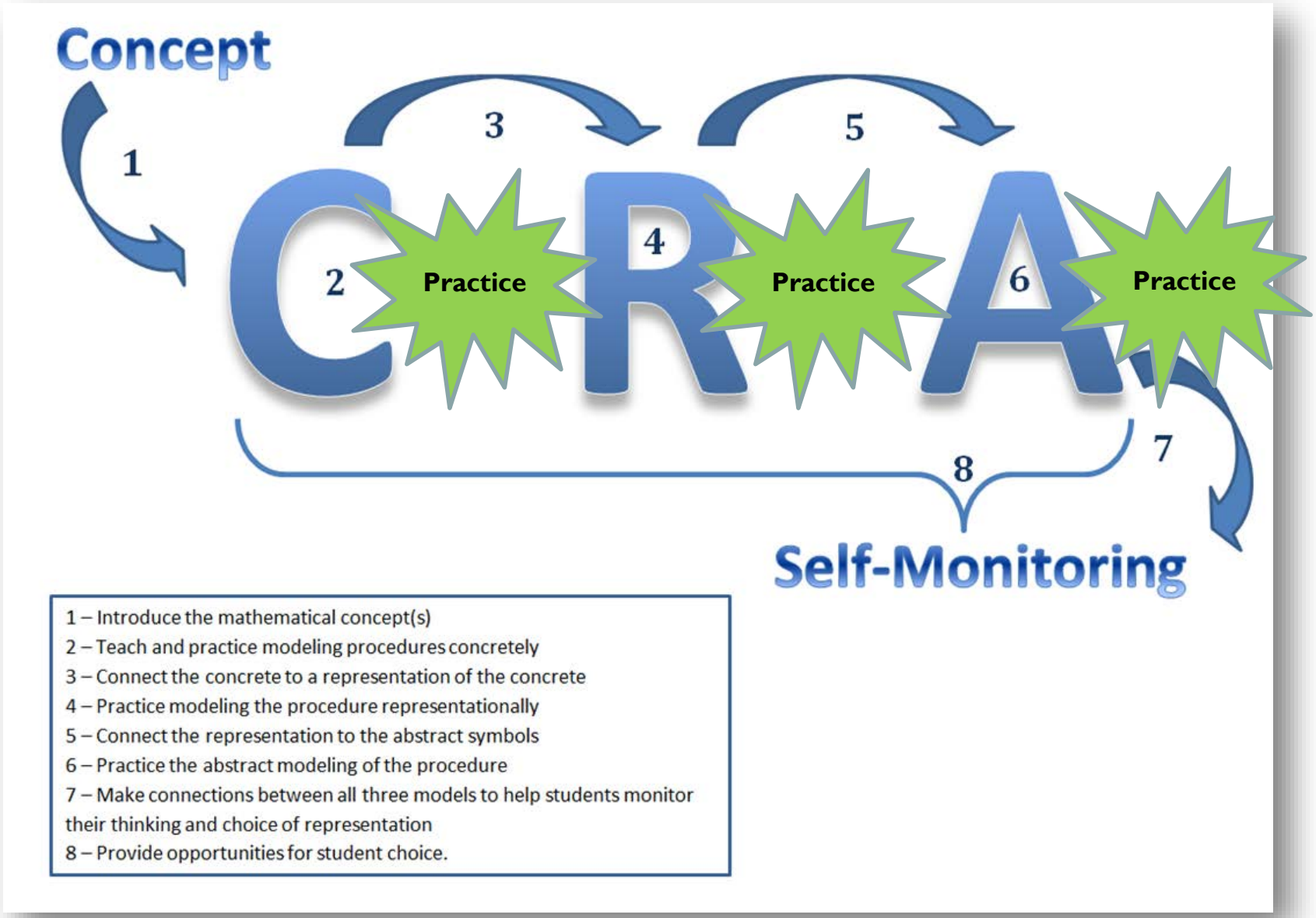
Teach the blocks!



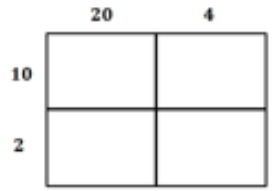
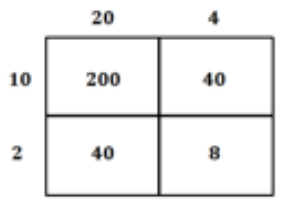
Teach the blocks!



CRA Sequence of Instruction

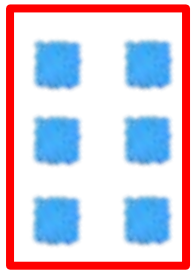


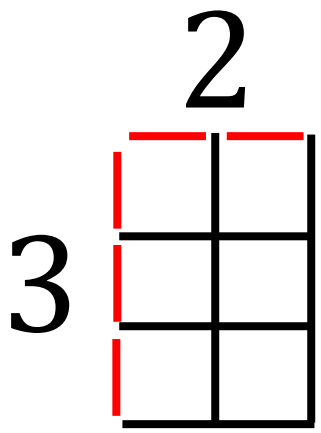
Multiplication (place value & partial products)

Abstract	Abstract	Language
$12 = 10 + 2$	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times \\ \hline \end{array}$	How can the first factor be written in expanded form?
$24 = 20 + 4$	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline \end{array}$	How can the second factor be written in expanded form?
	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \end{array}$	What is the product of the each number (digit) in the first factor by each number (digit) in the second factor
	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \\ 240 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \\ 200 + 40 \end{array}$	
"8 *ones*"	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \\ +240 \\ \hline 288 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \\ +200 + 40 \\ \hline 200 + 80 + 8 \end{array}$	How many ones does the product have?
no		Can I regroup 10 ones to make a ten?
"8 *tens*"		How many tens does the product have?
no		Can I regroup 10 tens to make a hundred?
"2 *hundreds*"		How many hundreds does the product have?
288	288	What is the final product?

Arrays & Area Models

3×2



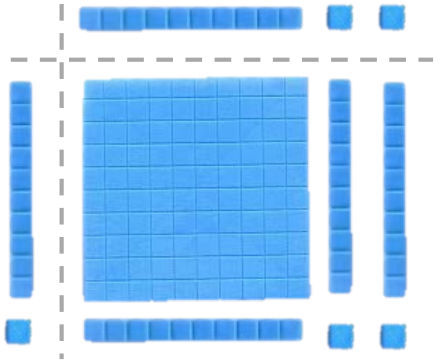


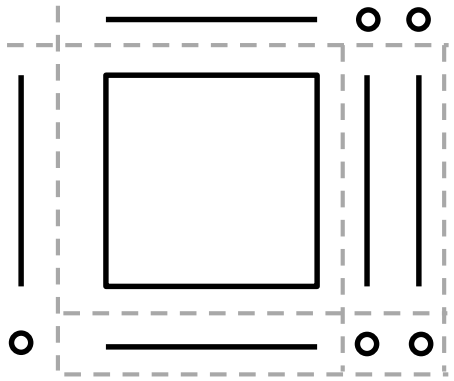
$$A = l * w$$

$$A = 2 \times 3$$

$$A = 6$$

Concrete – Representation – Abstract





$$\begin{array}{r} 12 \\ \times\ 11 \\ \hline 12 \\ +120 \\ \hline 132 \end{array}$$

	10	2	
10	100	20	= 120
1	10	2	= 12

Concrete/Representational Modeling

Partner Practice (model C/R, as teacher)

$$\begin{array}{r} 21 \\ \times\ 4 \\ \hline \end{array} \qquad \begin{array}{r} 22 \\ \times 13 \\ \hline \end{array} \qquad \begin{array}{r} 24 \\ \times 20 \\ \hline \end{array} \qquad \begin{array}{r} 17 \\ \times 11 \\ \hline \end{array}$$

$$\begin{array}{r} 23 \\ \times 12 \\ \hline \end{array} \qquad \begin{array}{r} 12 \\ \times 14 \\ \hline \end{array} \qquad \begin{array}{r} 13 \\ \times 32 \\ \hline \end{array} \qquad \begin{array}{r} 11 \\ \times 31 \\ \hline \end{array}$$

Connecting Models to the Abstract

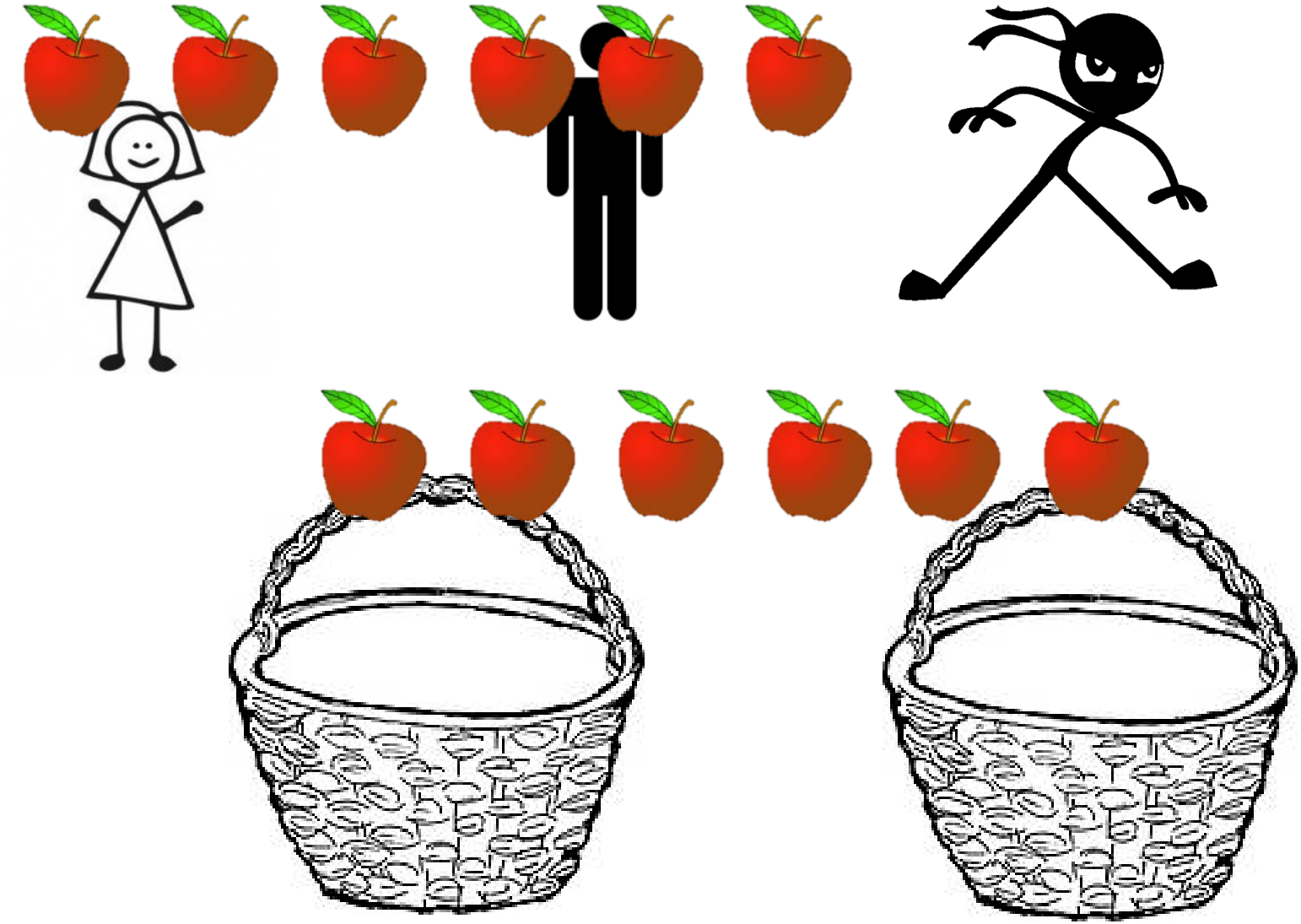
Partner Practice (model C/R to A)

$$\begin{array}{r} 26 \\ \times\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 40 \\ \times 32 \\ \hline \end{array} \qquad \begin{array}{r} 21 \\ \times 11 \\ \hline \end{array} \qquad \begin{array}{r} 10 \\ \times 30 \\ \hline \end{array}$$

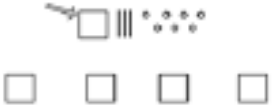
$$\begin{array}{r} 23 \\ \times 12 \\ \hline \end{array} \qquad \begin{array}{r} 31 \\ \times 16 \\ \hline \end{array} \qquad \begin{array}{r} 14 \\ \times 22 \\ \hline \end{array} \qquad \begin{array}{r} 39 \\ \times 47 \\ \hline \end{array}$$

Division

Early Division



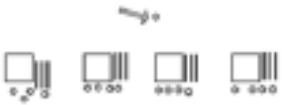
Division (place value & partial products)

Concrete	Representational	Language
		What is the number we are dividing? or What is the dividend? What are we dividing it into? or What is the divisor?
		How many hundreds are in each group?
		How many hundreds are remaining?
		How many tens is this worth?
		How many tens are in each group?

Division (place value & partial products)

Abstract (explicit trade algorithm)	Abstract (standard algorithm)	Language
$4\overline{)537}$	$4\overline{)537}$	What is the number we are dividing? or What is the dividend? What are we dividing it into? or What is the divisor?
$4\overline{)537}$	$4\overline{)537}$	How many hundreds are in each group?
$4\overline{)537}$ -4 1	$4\overline{)537}$ -4 1	How many hundreds are remaining?
$4\overline{)537}$ -4 13	$4\overline{)537}$ -4 13	How many tens is this worth?
$4\overline{)537}$ -4 13	$4\overline{)537}$ -4 13	How many tens are in each group?

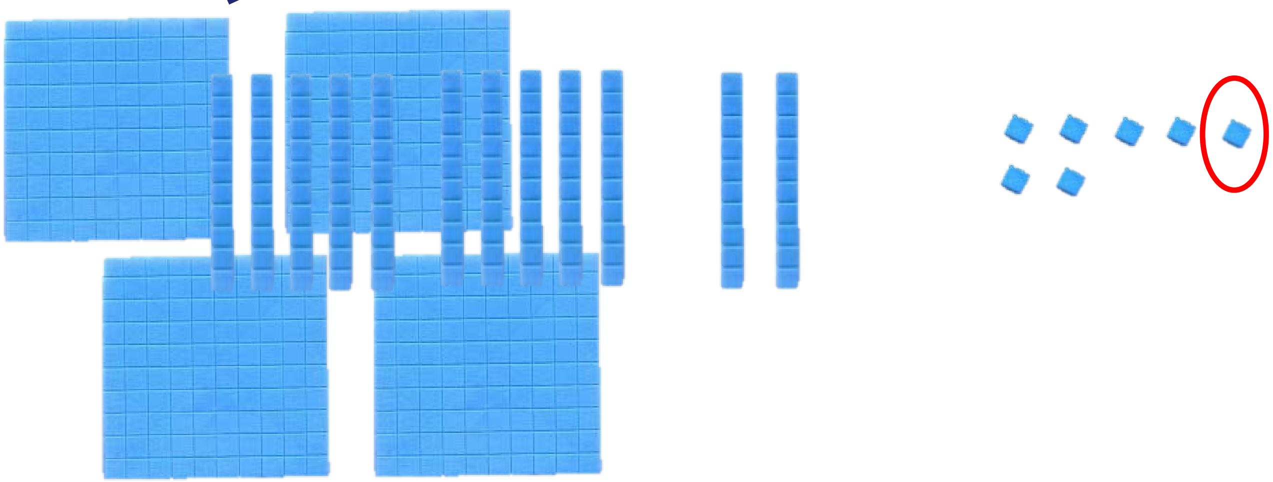
Division (place value & partial products)

		How many tens are remaining?
		How many ones is this worth?
		How many ones are in each group?
		How many ones are remaining?
$134\ R1$ *or* $134\frac{1}{4}$	$134\ R1$ *or* $134\frac{1}{4}$	What is the quotient? or How many are in one group with how many remaining?

Division (place value & partial products)

$4\overline{)537}$ -4 13 -12 1	$4\overline{)537}$ -4 13 -12 1	How many tens are remaining?
$4\overline{)537}$ -4 13 -12 17	$4\overline{)537}$ -4 13 -12 17	How many ones is this worth?
$4\overline{)537}$ -4 13 -12 17	$4\overline{)537}$ -4 13 -12 17	How many ones are in each group?
$4\overline{)537}$ -4 13 -12 17 -16 1	$4\overline{)537}$ -4 13 -12 17 -16 1	How many ones are remaining?
$4\overline{)537}\ R1$ = $4\overline{)537}$ -4 13 -12 17 -16 1	$4\overline{)537}\ R1$ = $4\overline{)537}$ -4 13 -12 17 -16 1	What is the quotient? or How many are in one group with how many remaining?

Concrete – Representation – Abstract



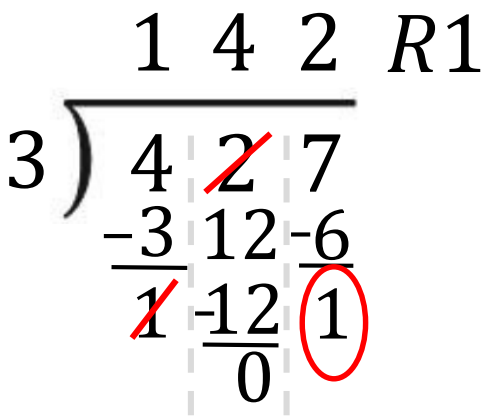
$$3\overline{)\,4\,2\,7}$$

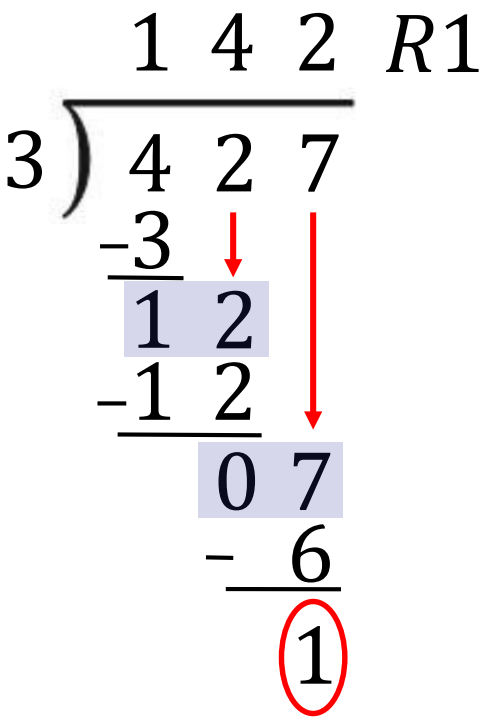
$$\begin{array}{r|r|r} 4 & \not{2} & 7 \\ -3 & 12 & -6 \\ \hline \not{1} & -12 & (1) \\ & 0 & \end{array}$$

$1\ 4\ 2\ R1$

or $1\ 4\ 2\ \frac{1}{3}$

Explicit Trade v. Traditional Algorithm





Why do we "bring down" the digits?

Partial quotients give partial remainders.

Partner Practice (model C/R, as teacher)

$3\overline{)62}$ $5\overline{)68}$ $4\overline{)71}$

$2\overline{)410}$ $4\overline{)202}$ $3\overline{)300}$

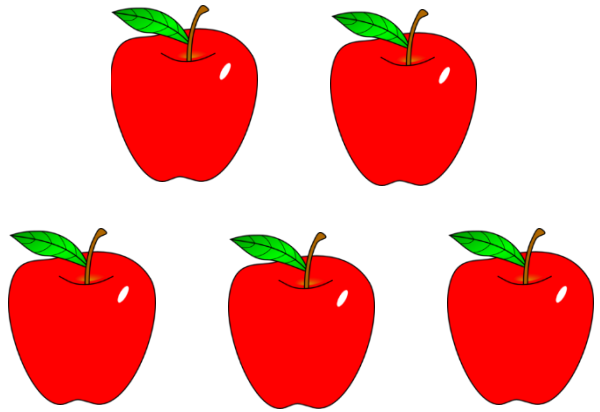
Fractions

Pennsylvania Training and Technical Assistance Network

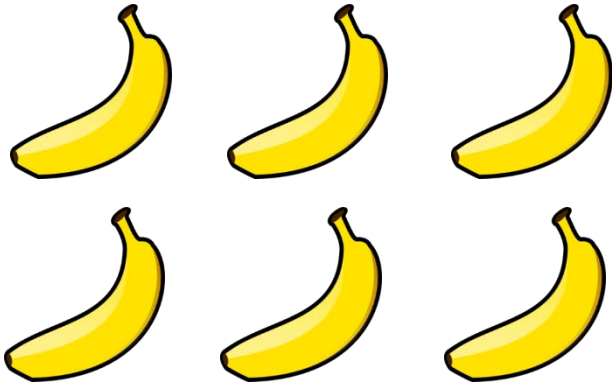
Fractions as "labels"

$2 + 3$	$2\ apples$ $+$ $3\ apples$	$2\ fourths$ $+$ $3\ fourths$	$\frac{2}{4} + \frac{3}{4}$
5	$5\ apples$	$5\ fourths$	$\frac{5}{4}$

Fractions as Labels



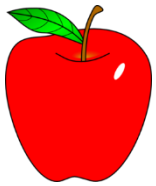
$$\frac{5}{\text{apple}}$$

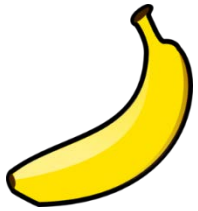


$$\frac{6}{\text{banana}}$$

Unit Fractions

$$\frac{a}{b} = a \times \frac{1}{b} \qquad counting\ "\ \frac{1}{b}'s\ "$$





$$\overline{3}$$

"$thirds$"


INTRAVERBAL

Unit Fractions

Measurement

$$\frac{a}{b} = a \times \frac{1}{b}$$

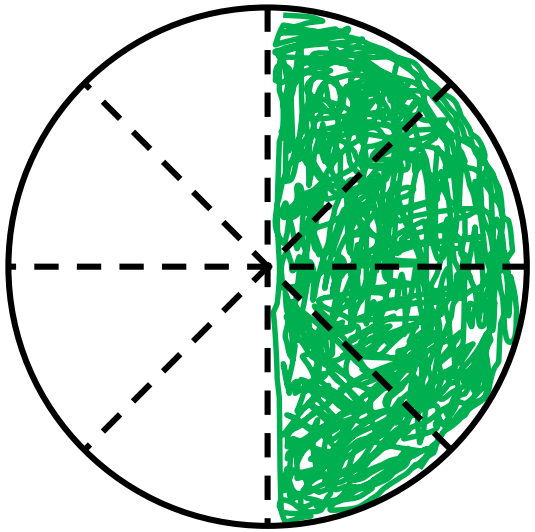
$$\frac{2}{3}\,\boldsymbol{cup}\ flour$$

$$= 2 \times \frac{1}{3}\,\boldsymbol{cup}\ flour$$

Unit Fractions

Geometry

$$\frac{a}{b} = a \times \frac{1}{b}$$



$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$$

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Bureau of Special Education
Pennsylvania Training and Technical Assistance Network

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Commonwealth of Pennsylvania

Tom Wolf, Governor