From Concrete to Representation to Abstract:

Math Instruction for Students with Autism

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Pennsylvania Training and Technical Assistance Network

PaTTAN's Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.

PDE's Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.

Session Outline

1. Introduction
2. Prerequisite Skills
3. **Place Value**
4. **Addition** **(& subtraction)**
5. **Multiplication** **(& division)**

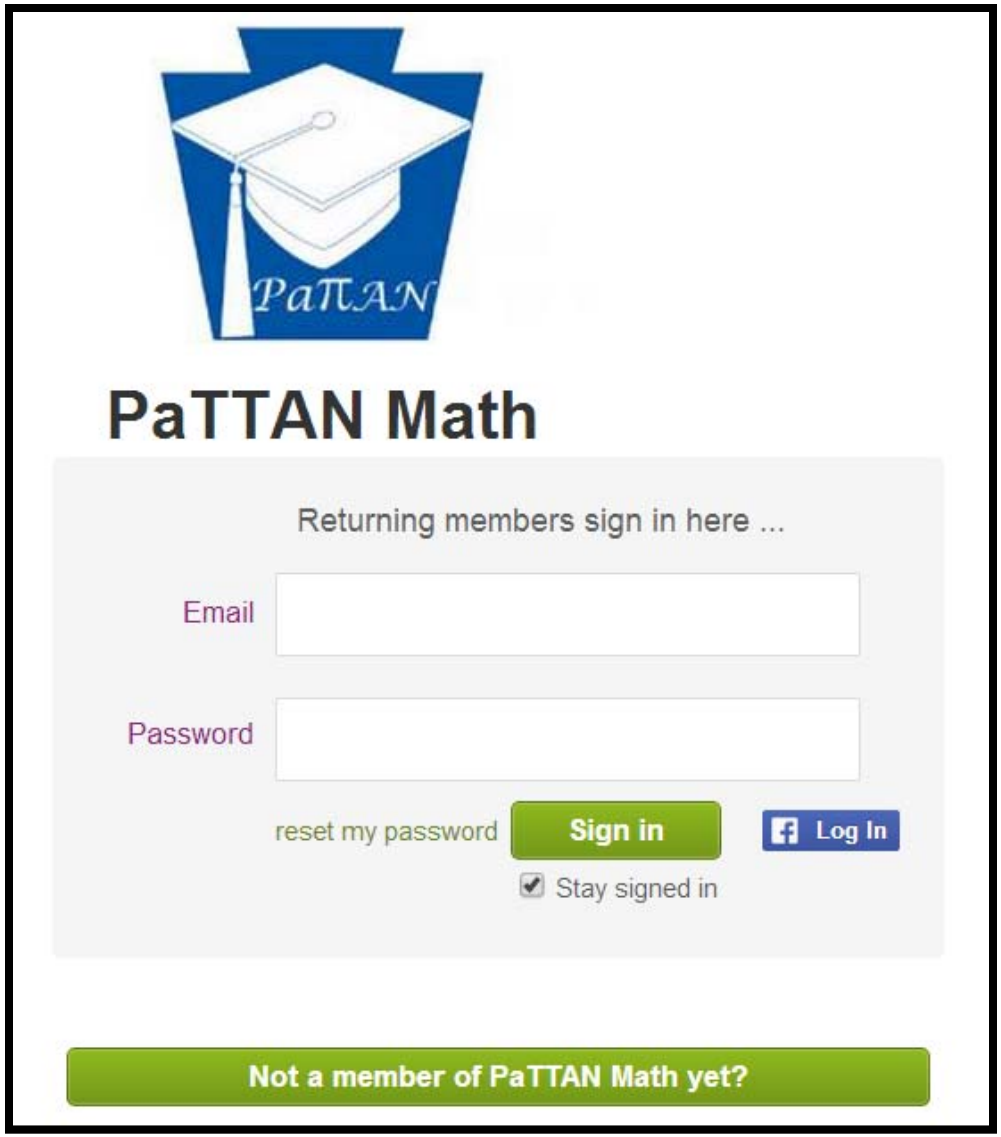
Preview CRA Progression

Demo Teaching (CR)

Group Practice (R)

CRA Map

6. Data Collection & Mastery Criteria



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CRA Math Instruction

FHN-J8M-HJ2

Where can I find more?

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PA Core Standards

<table>
<tr><th colspan="13">Mathematical Standards: Development and Progression</th></tr>
<tr><td colspan="13">Standards for Mathematical Practice
Make sense of problems and persevere in solving them.
Construct viable arguments and critique the reasoning of others.
Use appropriate tools strategically.
Look for and make use of structure.
Reason abstractly and quantitatively.
Model with mathematics.
Attend to precision.
Look for and express regularity in repeated reasoning.</td></tr>
<tr><td></td><td>PreK</td><td>K</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>HS</td></tr>
<tr><td rowspan="2">2.1 Numbers and Operations</td><td>(A) Counting & Cardinality</td><td colspan="11"></td></tr>
<tr><td></td><td colspan="5">(B) Numbers and Operations in Base Ten</td><td colspan="3">(D) Ratios and Proportional Relationships</td><td></td><td>(F) Number and Quantity</td></tr>
<tr><td></td><td colspan="4"></td><td colspan="3">(C) Numbers and Operations — Fractions</td><td colspan="3">(E) The Number System</td><td></td></tr>
<tr><td rowspan="2">2.2 Algebraic Concepts</td><td colspan="7">(A) Operations and Algebraic Thinking</td><td colspan="3">(B) Expressions and Equations</td><td>(D) Algebra</td></tr>
<tr><td colspan="9"></td><td colspan="3">(C) Functions</td></tr>
<tr><td>2.3 Geometry</td><td colspan="12">(A) Geometry</td></tr>
<tr><td>2.4 Measurement, Data, and Probability</td><td colspan="7">(A) Measurement and Data</td><td colspan="5">(B) Statistics and Probability</td></tr>
</table>

http://www.pdesas.org/Standard/PACore

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them


2. Reason abstractly and quantitatively


3. Construct viable arguments and critique the reasoning of others


4. Model with mathematics

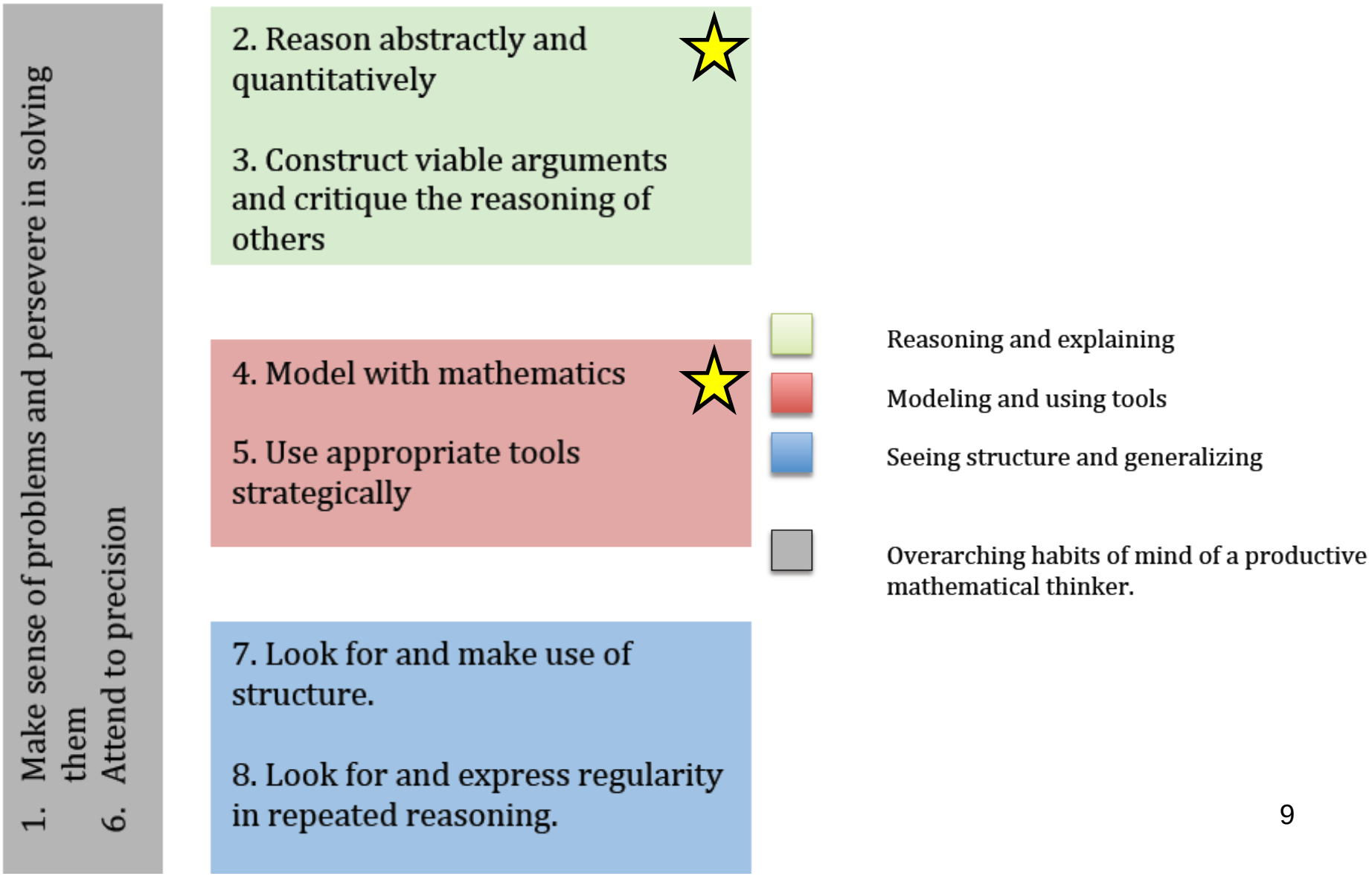

5. Use appropriate tools strategically

6. Attend to precision

7. Look for and make use of structure

8. Look for and express regularity in repeated reasoning

Mathematical Practices Structure Diagram



Rationale – IES Practice Guide

Intervention materials should include opportunities for students to work with **visual representations** of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.

Evidence: **Moderate**
Source: IES Practice Guide: Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools

Instruction during the intervention should be **explicit and systematic**. This includes providing models of proficient problem solving, **verbalization** of thought processes, guided practice, corrective feedback, and frequent cumulative review.

Evidence: **Strong**
Source: IES Practice Guide: Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools

Help students understand **why procedures for computations** with fractions make sense.

Evidence: **Moderate**
Source: IES Practice Guide: Developing Effective Fractions Instruction for Kindergarten Through 8th Grade

Rationale – Additional Research

Research-based studies show that students who use **concrete materials** develop more **precise** and more **comprehensive** mental representations, often show more motivation and on-task behavior, understand mathematical ideas, and better apply these ideas to life situations (Harrison & Harrison, 1986; Suydam & Higgins, 1977).

Harrison, M., & Harrison, B. (1986). Developing numeration concepts and skills. Arithmetic Teacher, 33, 1–21.

Suydam, M. N.; & Higgins, J. L. (1977). *Activity-based learning in elementary school mathematics: Recommendations from research*. Columbus, OH: ERIC Center for Science, Mathematics, and Environmental Education.

Prerequisites

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Make sure students have necessary skills to begin CRA.

A math placement test does not necessarily give you this information.

We will not cover all requisite skills that might need to be addressed in a students programming prior to using CRA.

Individualized programming decisions will be necessary, these decisions should be based on student data (rates of mastery, frequency of errors, fluent responding, etc.).

There are several conference sessions addressing instructional basics that will provide valuable information.

Group Instruction

Respond in group to known instructions (unison/choral responding as well as responding to single responses when appropriate, responding to hand signals/cues)

Learn new behaviors/master new skills in a group format

Work independently in a group for 5-15 minutes and stay on task

Follow instructions involving adjectives and prepositions (Put the ones block under the tens block.)

Follow instructions regarding pronouns and adverbs (My turn. Okay, now your turn. Draw an area model then draw a tens block.)

Prerequisites for Beginning Math Instruction for Students with Autism

Label novel objects/text
(long vs. short, big vs. little, able to read text written vs. printed, etc.)

Echo novel phrases

Match novel objects

Imitate novel motor movements

Receptively identify objects/follow directions
(area map, choral response, etc.)

Answers questions flexibly

Fluent responding – A skill is really learned when the student can do it fluently. Fluency means the student can do the skill accurately and quickly. (e.g. Subitization)

If a student is not able to master these skills over a few teaching trials it is likely that instruction should focus on teaching a more developed verbal repertoire before beginning CRA.

If essential precursor skills have not been mastered, it is unlikely that the student will learn CRA in a meaningful way (generalization).

If an assessment of the student's language indicates that specific training is necessary, it is essential that instruction first focus on prerequisite verbal skills.

Mathematics requires a student to engage in complex verbal behavior. (Concepts of numbers – 3 is the same as III which is equal to three which is equivalent to ●●●. "4 and 3 is 7" is the same as 4+3=7.)

Mathematics is verbal behavior, how we define mathematical concepts requires clear social communication: to teach math effectively you must teach communication.

Place Value

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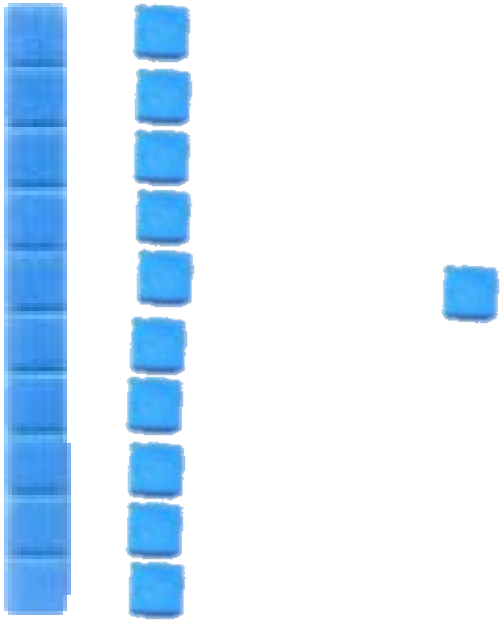
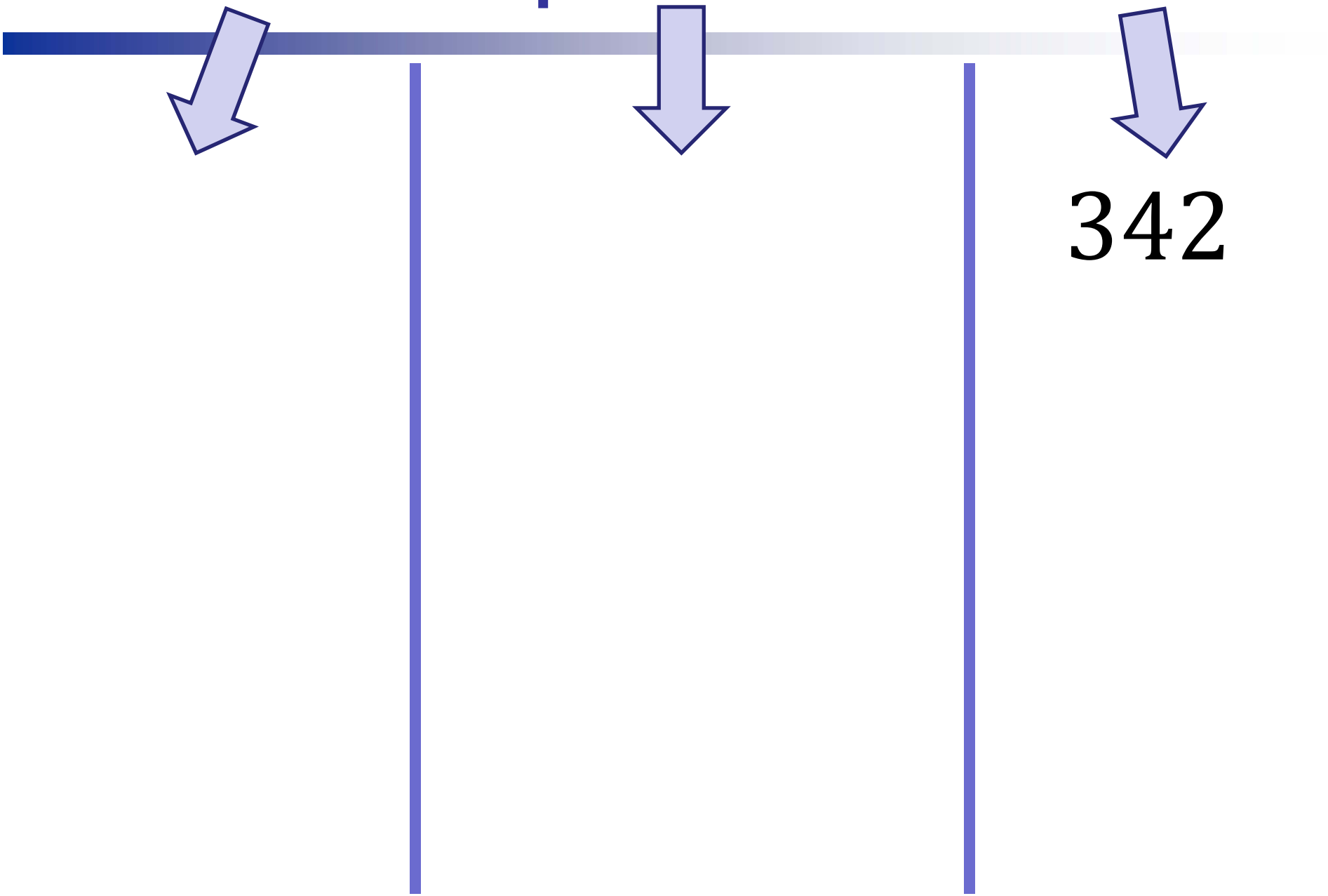
Teach the blocks!



Concrete – Representation – Abstract



Place Value – Problem #1

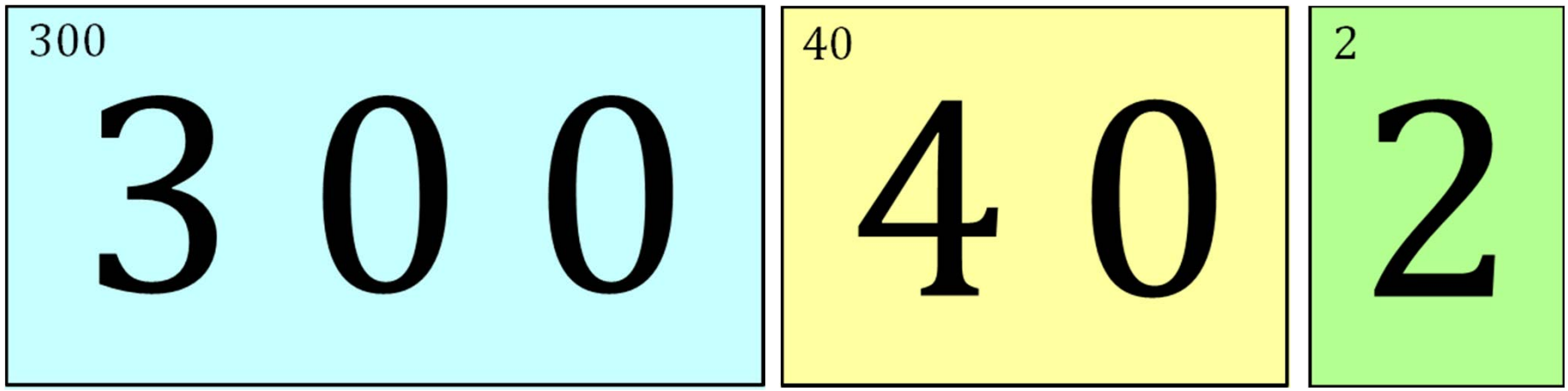
Place Value – Problem #2

Place Value – Problem #3

Place Value – Problem #4

Place Value Cards

342



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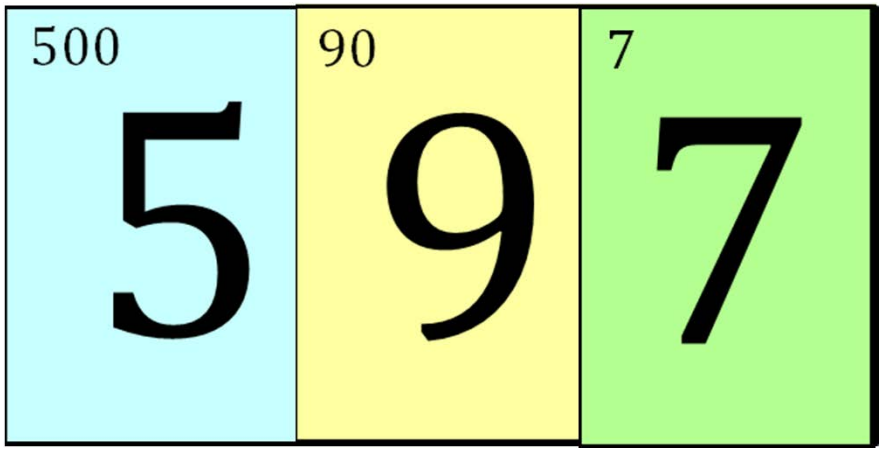
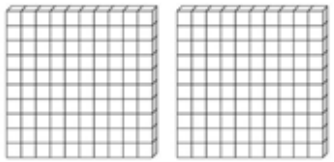
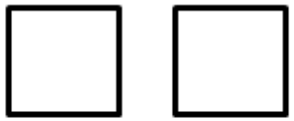
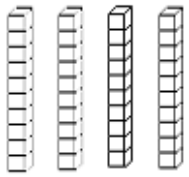
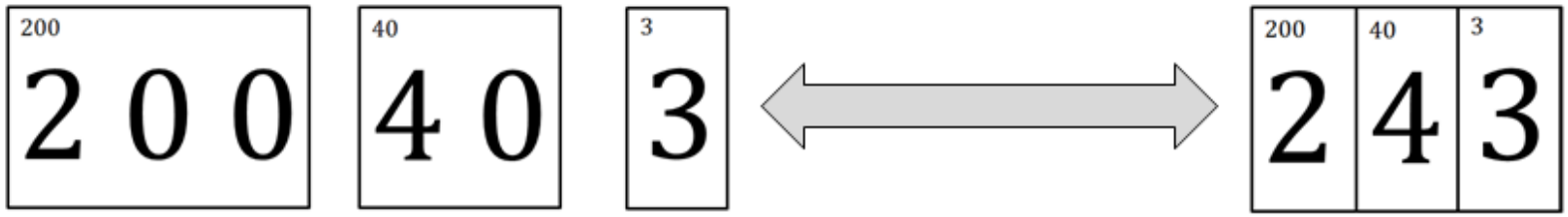


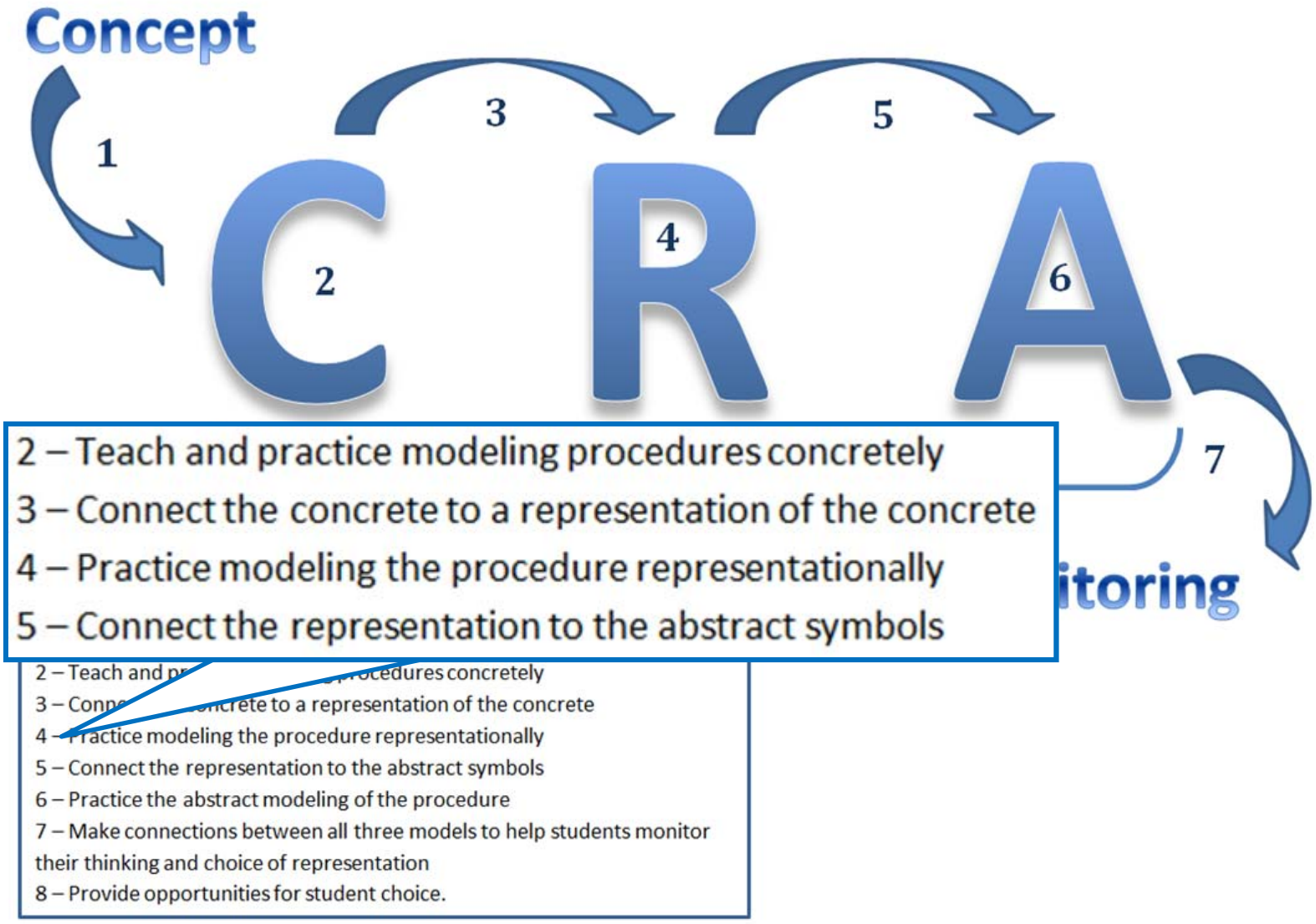
Place Value (expanded form)

Concrete	Representational	Abstract	Language
		243; 2 *hundreds* 200	How many hundreds? What is that worth?
		243; 4 *tens* 40	How many tens? What is that worth?
		243; 3 *ones* 3	How many ones? What is that worth?
200 + 40 + 3 (using place value mat)	200 + 40 + 3	200 + 40 + 3	How do we write the number in expanded form?
"two hundred forty three"	*"two hundred forty three"*	*"two hundred forty three"*	How do we read the number?

May support the Abstract portions with place value cards



CRA Sequence of Instruction



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Addition

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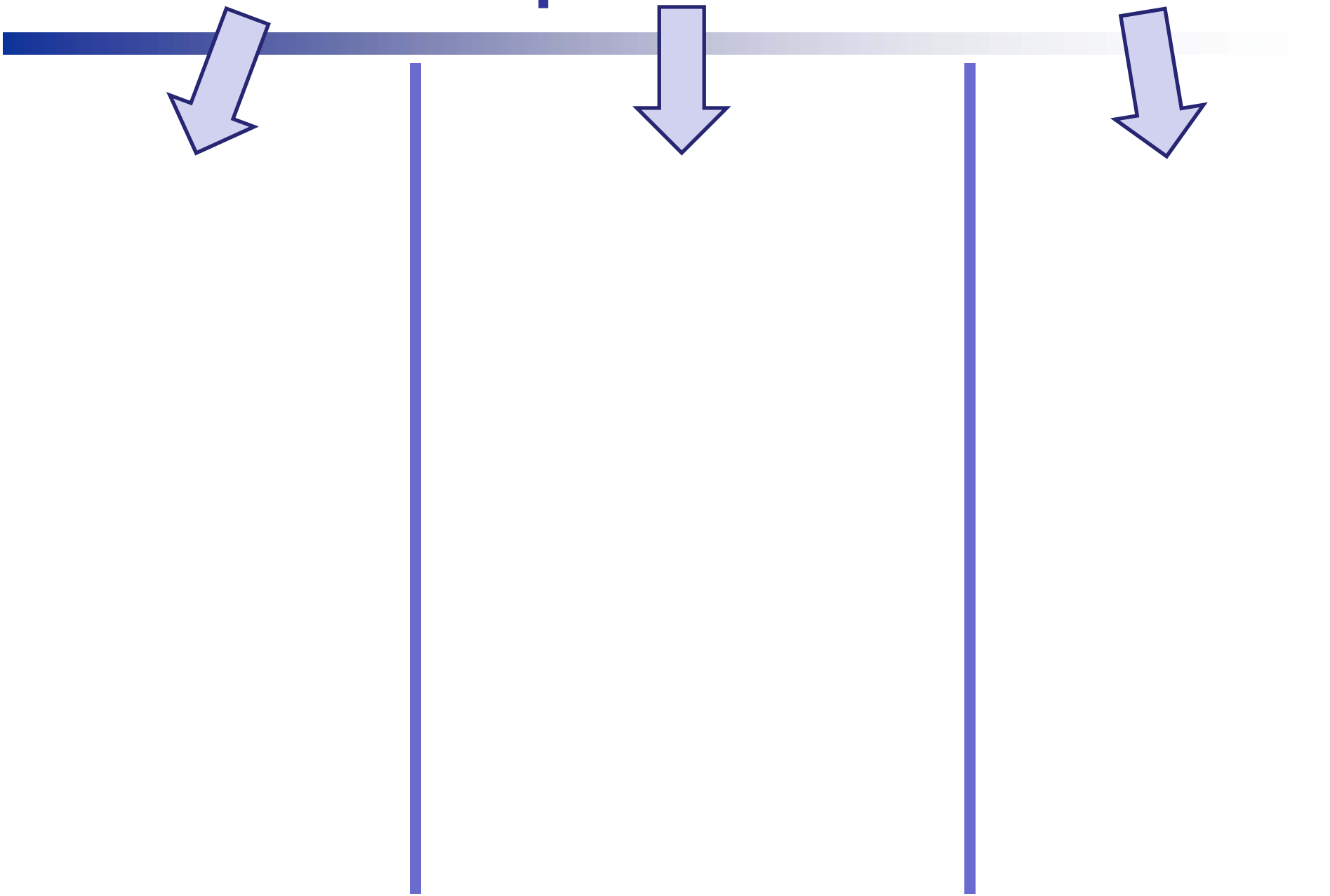
Concrete – Representation – Abstract

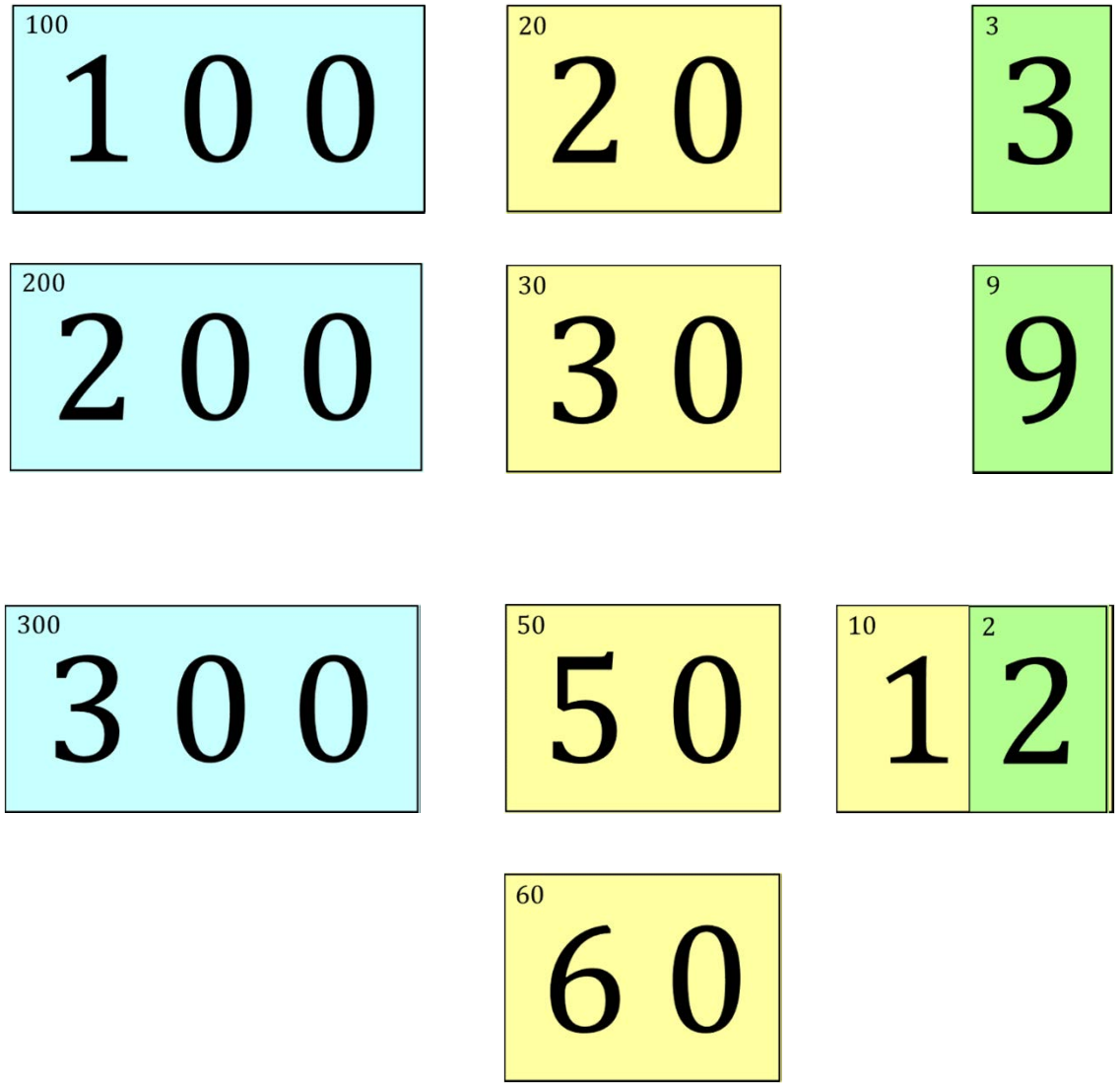


Addition – Problem #1

Addition – Problem #2

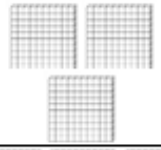
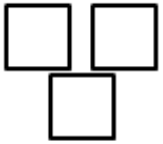
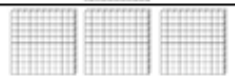
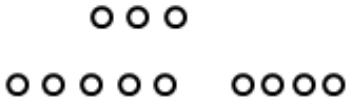
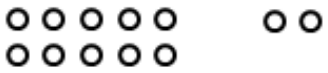
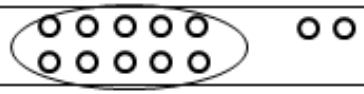
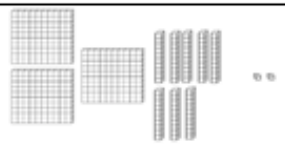
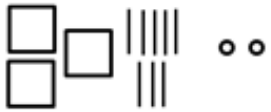
Addition – Problem #4

Addition – Problem #5 *(place value cards)*



$$\begin{array}{r} 123 \\ +239 \\ \hline 300 \\ 50 \\ +\ 12 \\ \hline 362 \end{array}$$

Addition (place value & partial sums)

Concrete	Representational	Abstract	Language
Addition			
		243 +139	How many hundreds does each number have?
 *"3 hundreds"*	 *"3 hundreds"*	243 +139 300	How many hundreds does the sum have?
		243 +139 300	How many tens does each number have?
 *"7 tens"*	 *"7 tens"*	243 +139 300 70	How many tens does the sum have?
		243 +139 300 70	How many ones does each number have?
 *"12 ones"*	 *"12 ones"*	243 +139 300 70 12	How many ones does the sum have?
		243 +139 300 70 12 382	Can I regroup 10 ones to make a ten?
No	**No**		Can I regroup 10 tens to make a hundred?
			What is the sum of #1 and #2?
After partial sums is understood, you can generalize the algorithm from right to left.			

Subtraction

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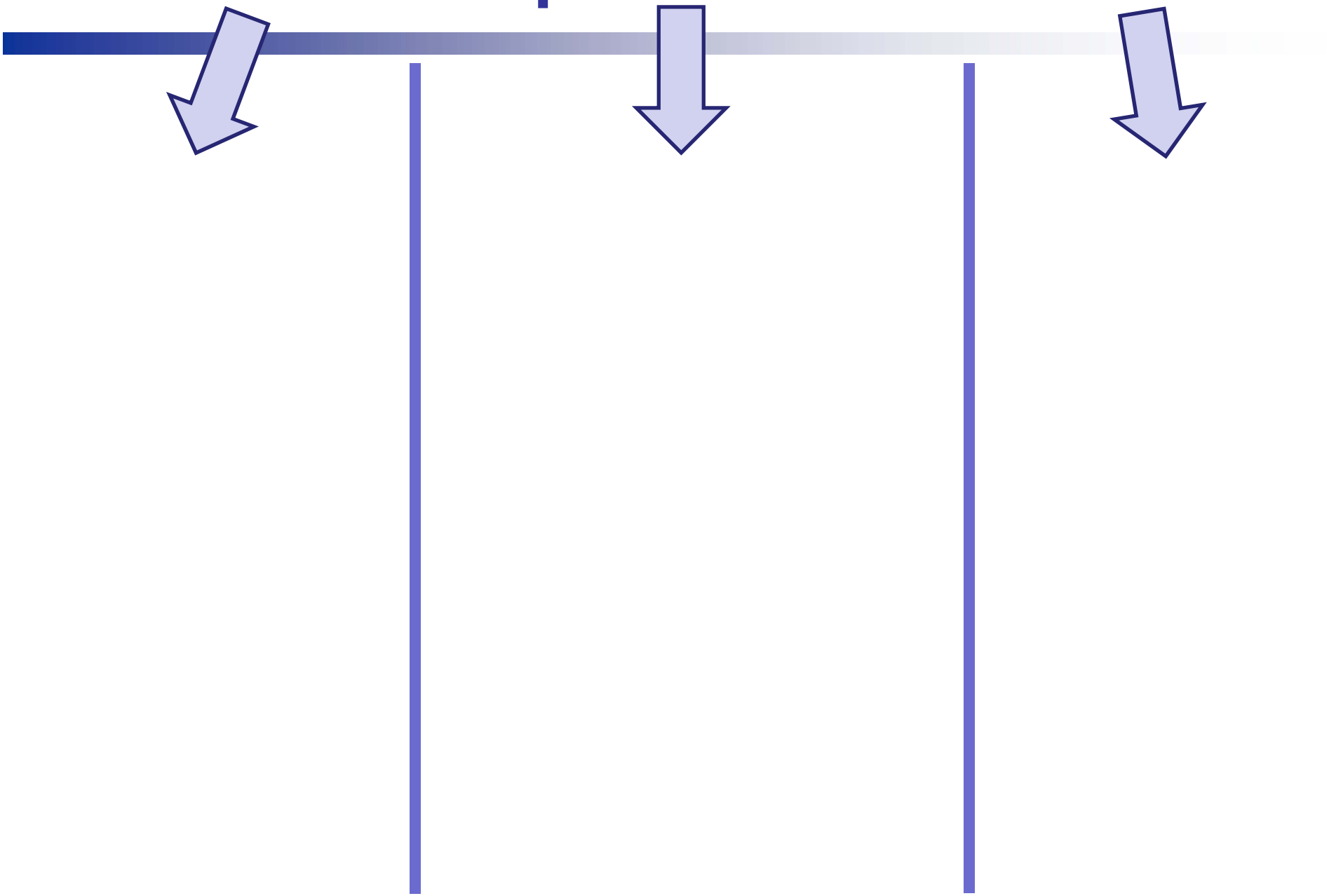
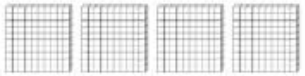
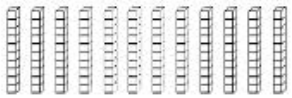
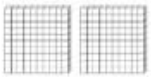
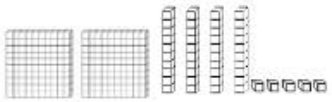
Concrete – Representation – Abstract



Subtraction (place value)

Concrete	Representational	Abstract	Language
		$\begin{array}{r} 423 \\ -178 \end{array}$	How many hundreds does the minuend have?
		$\begin{array}{r} 423 \\ -178 \end{array}$	How many tens does the minuend have?
		$\begin{array}{r} {\scriptstyle 3\ 12} \\ \not{4}\not{2}3 \\ -178 \end{array}$	Can I complete the subtraction of tens *without using negatives*, or do I need to make 10 tens from a hundred?
		$\begin{array}{r} {\scriptstyle 3\ 12} \\ \not{4}\not{2}3 \\ -178 \end{array}$	How many ones does the minuend have?
		$\begin{array}{r} {\scriptstyle 11} \\ {\scriptstyle 3\ \not{1}\not{2}\ 13} \\ \not{4}\not{2}\not{3} \\ -178 \end{array}$	Can I complete the subtraction of ones *without using negatives*, or do I need to make 10 ones from a ten?
		$\begin{array}{r} {\scriptstyle 11} \\ {\scriptstyle 3\ \not{1}\not{2}\ 13} \\ \not{4}\not{2}\not{3} \\ -178 \\ \hline 5 \end{array}$	What is the difference between the ones?
		$\begin{array}{r} {\scriptstyle 11} \\ {\scriptstyle 3\ \not{1}\not{2}\ 13} \\ \not{4}\not{2}\not{3} \\ -178 \\ \hline 45 \end{array}$	What is the difference between the tens?
		$\begin{array}{r} {\scriptstyle 11} \\ {\scriptstyle 3\ \not{1}\not{2}\ 13} \\ \not{4}\not{2}\not{3} \\ -178 \\ \hline 245 \end{array}$	What is the difference between the hundreds?
		245	What is the difference between #1 and #2? *or* What is #1 minus #2?

Where can I find more?

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Multiplication

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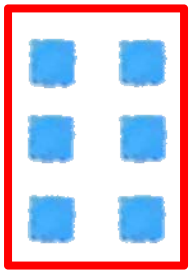
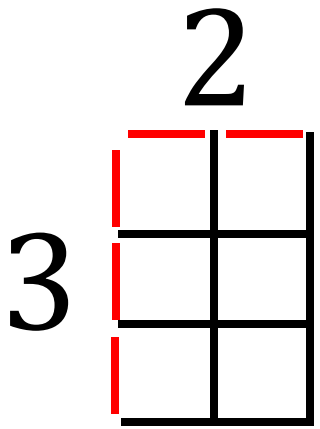
Arrays & Area Models

3×2



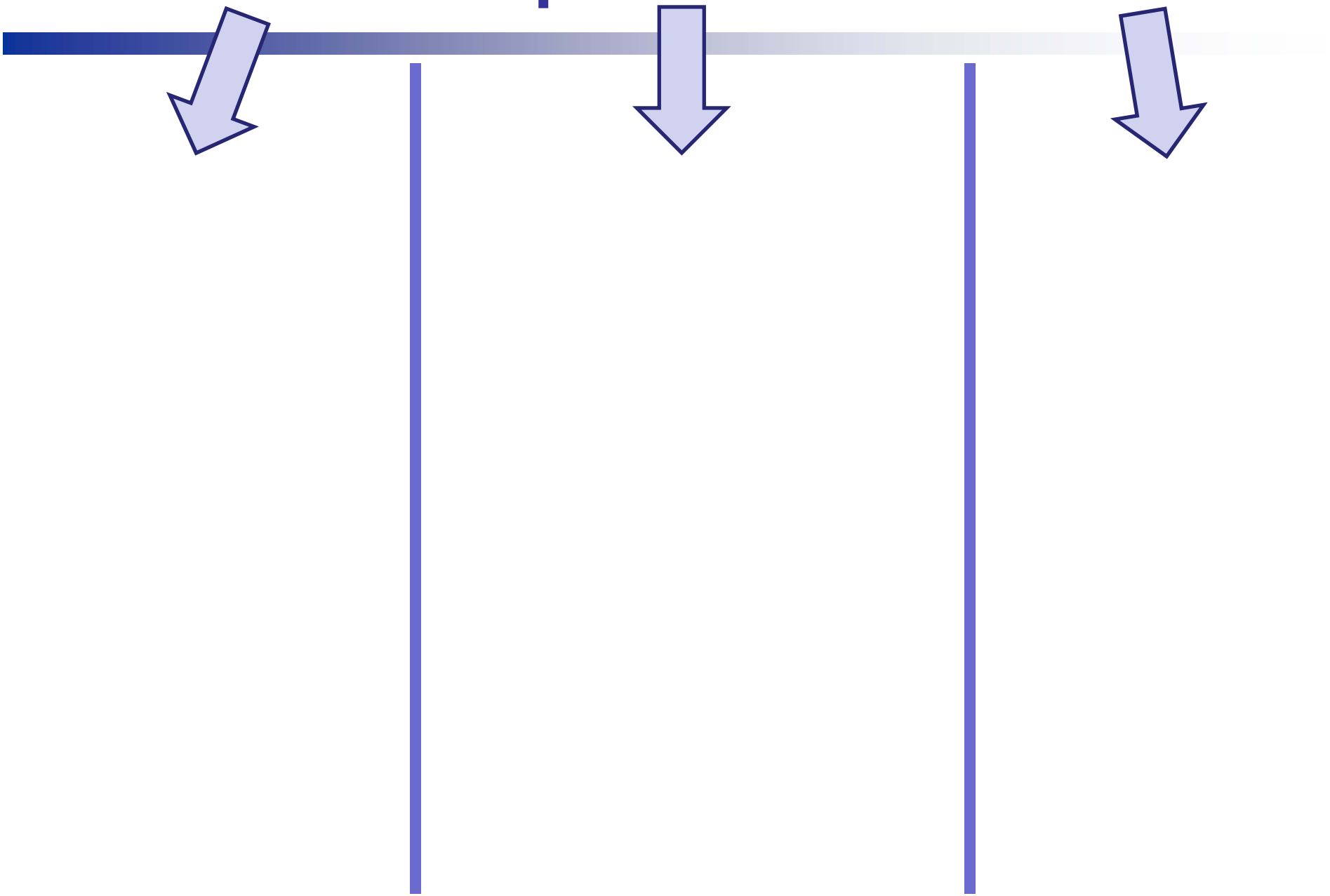


$$A = l * w$$

$$A = 2 \times 3$$

$$A = 6$$

Concrete – Representation – Abstract



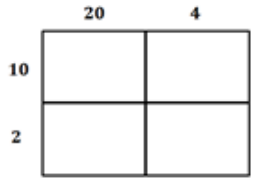
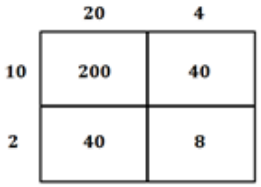
Multiplication – Problem #1

Multiplication – Problem #2

Multiplication – Problem #3

Multiplication – Problem #3. 1

Multiplication (place value & partial products)

Representational	Abstract	Language
$12 = 10 + 2$	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times \\ \hline \end{array}$	How can the first factor be written in expanded form?
$24 = 20 + 4$	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline \end{array}$	How can the second factor be written in expanded form?
	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \end{array}$	What is the product of the each number (digit) in the first factor by each number (digit) in the second factor
	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \\ 240 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \\ 200 + 40 \end{array}$	
"8 *ones*"	$\begin{array}{r} 12 \\ \times\ 24 \\ \hline 48 \\ +240 \\ \hline 288 \end{array}$ $\qquad$ $\begin{array}{r} 10 + 2 \\ \times\ 20 + 4 \\ \hline 40 + 8 \\ +200 + 40 \\ \hline 200 + 80 + 8 \end{array}$	How many ones does the product have?
no		Can I regroup 10 ones to make a ten?
"8 *tens*"		How many tens does the product have?
no		Can I regroup 10 tens to make a hundred?
"2 *hundreds*"		How many hundreds does the product have?
288	288	What is the final product?

Where can I find more?

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Division

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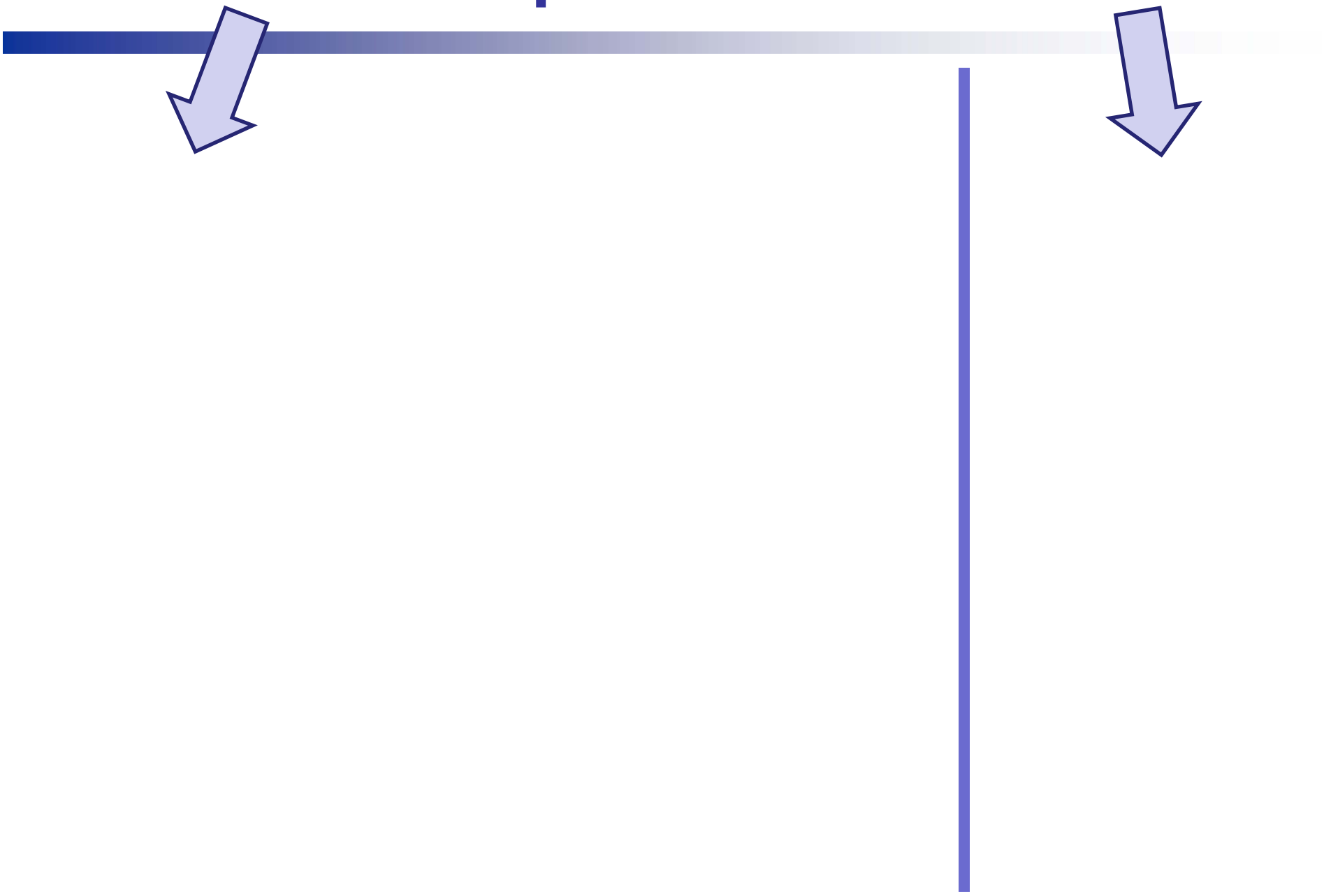
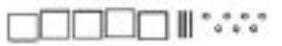
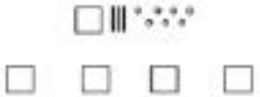
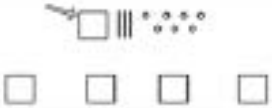
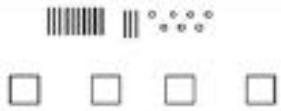
Concrete – Representation – Abstract



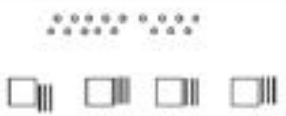
Division (place value & partial products)

Concrete	Representational	Language
		What is the number we are dividing? or What is the dividend? What are we dividing it into? or What is the divisor?
		How many hundred are in each group?
		How many hundreds are remaining?
		How many tens is this worth?
		How many tens are in each group?

Division (place value & partial products)

Abstract (explicit/crude algorithm)	Abstract (standard algorithm)	Language
$4\overline{)537}$	$4\overline{)537}$	What is the number we are dividing? or What is the dividend? What are we dividing it into? or What is the divisor?
$4\overline{)537}$ 1	$4\overline{)537}$ 1	How many hundred are in each group?
$4\overline{)537}$ -4 1	$4\overline{)537}$ -4 1	How many hundreds are remaining?
$4\overline{)537}$ -4 13	$4\overline{)537}$ -4 13	How many tens is this worth?
$4\overline{)537}$ -4 13 1	$4\overline{)537}$ -4 13	How many tens are in each group?

Division (place value & partial products)

		How many tens are remaining?
		How many ones is this worth?
		How many ones are in each group?
		How many ones are remaining?
 $134\,R1$ or $134\frac{1}{4}$	 $134\,R1$ or $134\frac{1}{4}$	What is the quotient? or How many are in one group with how many remaining?

Division (place value & partial products)

$4\overline{)537}$ -4 13 -12 1	$4\overline{)537}$ -4 13 -12 1	How many tens are remaining?
$4\overline{)537}$ -4 13 -12 17 -16 1	$4\overline{)537}$ -4 13 -12 17 -16 1	How many ones is this worth?
$4\overline{)537}$ -4 13 -12 17 -16 1	$4\overline{)537}$ -4 13 -12 17 -16 1	How many ones are in each group?
$4\overline{)537}$ -4 13 -12 17 -16 1	$4\overline{)537}$ -4 13 -12 17 -16 1	How many ones are remaining?
$4\overline{)537}$ -4 13 -12 17 -16 1 $R1$ = $4\overline{)537}$ -4 13 -12 17 -16 1 $\frac{1}{4}$	$4\overline{)537}$ -4 13 -12 17 -16 1 $R1$ = $4\overline{)537}$ -4 13 -12 17 -16 1 $\frac{1}{4}$	What is the quotient? or How many are in one group with how many remaining?

Language
What is the number we are dividing? or What is the dividend? What are we dividing it into? or What is the divisor?
How many hundred are in each group?
How many hundreds are remaining?
How many tens is this worth?
How many tens are in each group?

How many tens are remaining?
How many ones is this worth?
How many ones are in each group?
How many ones are remaining?
What is the quotient? or How many are in one group with how many remaining?

Data Collection & Mastery Criteria

Pennsylvania Training and Technical Assistance Network

Data Collection

Considerations:

- Acquisition – Cold probe, permanent product
- Fluency – Time trials
- Application – Practice measures

Language Structure

$a \pm b$

Everyone stand on zero and face the positive direction.

The first number is _____ (a). Everyone move _____ ($|a|$) numbers _____.

"forward" (if positive)

"backward" (if negative)

The sign says to _____ (*"add"* / *"subtract"*), so everyone face the _____ (*"positive"* / *"negative"*) direction.

The second number is _____ (b). Everyone move _____ ($|b|$) numbers _____.

"forward" (if positive)

"backward" (if negative)

What number are you standing on?

Teacher writes solution. Teacher reads solutions. Students read solution

Data Collection

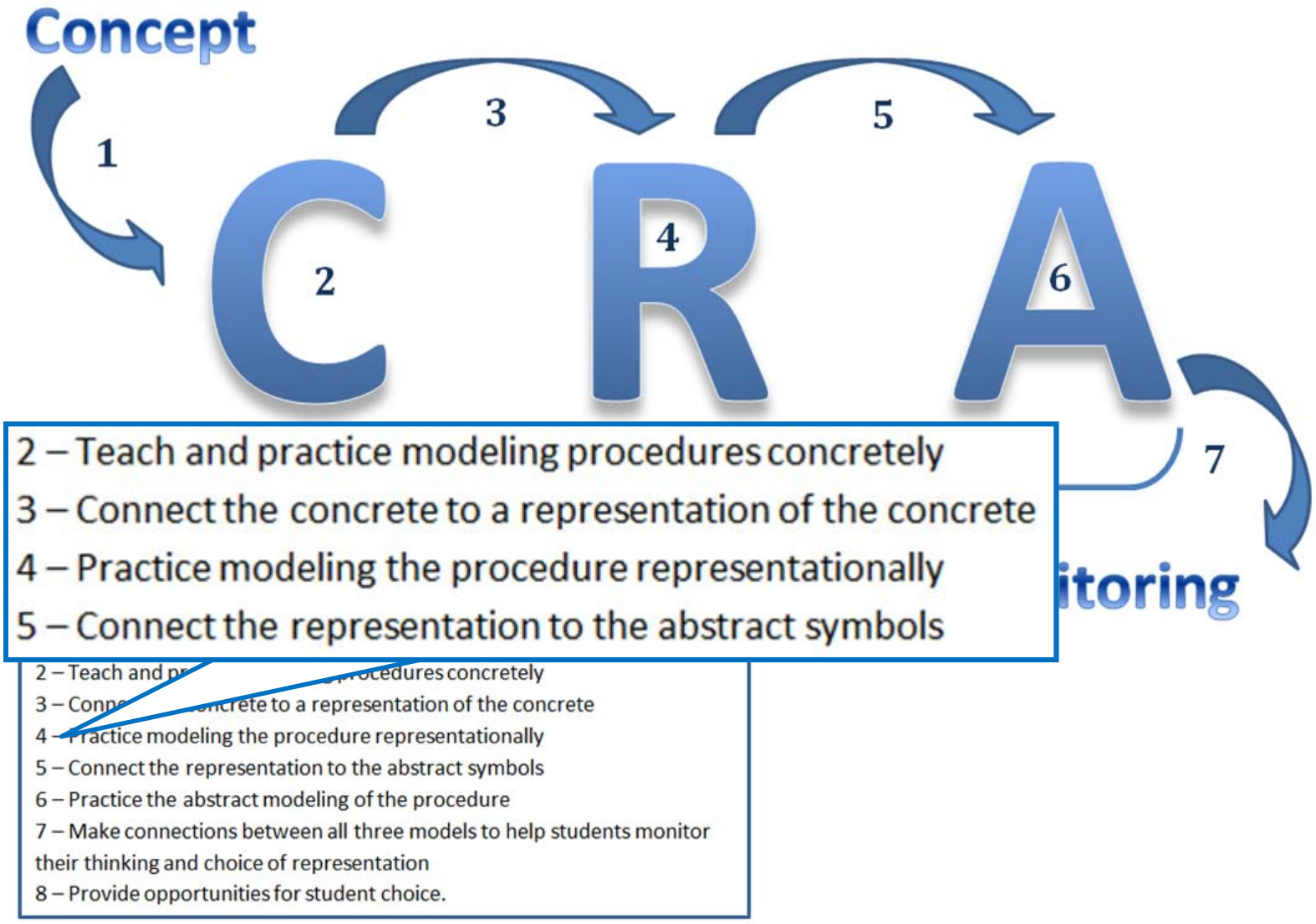
[redacted]

Adding/Subtracting Integers Data Log

Problem	Name	Zero/ Face positive	Locate first number	Operation/ Directional change	Second number shift	Determine location	Read completed number sentence
5+2	[redacted]	X	X	X	X	X	X
		X	X	/	X	X	X
3+3		X	X	X	X	X	X
		X	X	X	X	X	X
8-3		X	X	X	X	X	X
		X	/	X	X	X	X
9-2		X	X	X	X	X	X
		X	X	X	X	X	X
5+-2		X	X	X	X	X	X
		X	/	X	X	X	X
8+-3		X	X	X	X	X	X
		X	X	X	X	X	X
2+-3		X	X	X	X	X	X
		X	X	X	X	X	X
1+-4		X	X	X	X	X	X
		X	X	X	X	X	X
4+-4		X	X	X	X	X	X
		X	X	X	X	X	X
3+-1		X	X	X	X	X	X
		X	X	X	X	X	X

X – student(s) complete independently **/** – student(s) complete with support **Leave blank** – student(s) not able to complete

CRA Sequence of Instruction



Where can I find more?

PaTTAN Math

#ptnmath

www.pattan.net

www.pattanmath.wiggio.com

Pennsylvania Training and Technical Assistance Network

Contact Information www.pattan.net

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Patricia Hozella, Director
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